Quasi-Linear Integrability*

Volker Nocke† Nicolas Schutz‡

March 17, 2017

Abstract

Applied researchers often work with demand systems that do not depend on income, with the implicit assumption that preferences are quasi-linear and income sufficiently large. The classic approach to the integrability of demand does not readily apply in this case. Adopting a much simpler approach that is based on integrating the vector field defined by the demand system and on duality, we provide necessary and sufficient conditions for the quasi-linear integrability of such (continuous) demand systems. We also derive results on the associated utility function and its domain, and provide an application to the analysis of demand systems in the presence of measurement errors.

1 Introduction

In this paper, we provide necessary and sufficient conditions for the quasi-linear integrability of a demand system. That is, we analyze the classic *integrability problem* for the case when observed demand does not vary with income and consumers have quasi-linear preferences.

In industrial organization and other applied fields of microeconomics, researchers typically focus on partial equilibrium settings and therefore often work with demand systems that depend on prices but not on income. The implicit assumption they make is that preferences are quasi-linear and consumer income sufficiently high, so that not all income is spent on the goods offered in the market under consideration. It remains unspecified what demand would look like if income were so low that no outside good would be consumed.

The classic treatment of the integrability problem is due to Hurwicz and Uzawa (1971). Their approach consists of two key steps. First, they integrate the demand system to obtain an income compensation function, using tools from the literature on differential forms. Second, inverting the demand system and using the income compensation function, they construct a utility function, defined over the range of the demand system. The second step has

*We thank the editor, two anonymous referees, and John Quah for helpful comments. Nocke gratefully acknowledges financial support from the European Research Council (project no. 313623).

†UCLA, University of Mannheim, and CEPR. Email: volker.nocke@gmail.com.

‡University of Mannheim. Email: schutz@uni-mannheim.de.

been subsequently improved by Jackson (1986), who uses duality theory to obtain an upper semi-continuous utility function defined over the entire non-negative orthant. However, this approach does not apply (and does not extend easily) to the "quasi-linear case" in which demand is not a function of income. In particular, the existing approach relies on (i) the differentiability of demand at all prices and income levels, (ii) the boundary condition that, for any price vector, demand is zero whenever income is zero, and (iii) the non-negativity of demand.

How can the existing approach and results be extended to the quasi-linear case in which the specified demand is not a function of income?[1] One possible solution would involve allowing for negative consumption of the outside good whenever income is sufficiently low. However, this would obviously violate conditions (ii) and (iii). An alternative solution would involve extending the demand system in some way to account for cases in which income is sufficiently low. However, this would necessarily introduce a non-differentiability, violating condition (i).[2]

In this paper, we develop a much simpler approach for continuous demand systems that do not depend on income. First, we show the existence of a (candidate) indirect subutility function by integrating the vector field defined by the demand system. Second, we construct a direct quasi-linear utility function, using duality theory. We show that quasi-linear integrability amounts to the demand system being a conservative vector field that satisfies the law of demand. If demand is continuously differentiable, then this condition is equivalent to the symmetry and negative semi-definiteness of the substitution matrix of the demand system.[3] Moreover, under these conditions, the associated indirect utility function is unique up to an additive constant, and continuously differentiable, implying that any induced change in money-metric consumer welfare is uniquely pinned down. Under the same conditions, the demand system can be derived from a monotone, concave and upper semi-continuous subutility function defined over some set X that contains the comprehensive convex hull of the range of the demand system. Furthermore, the resulting subutility function is continuous in the interior of X.

We obtain additional results in Section 4. There, we provide results on the shape of X, on the maximality of X, derive conditions under which the subutility function is continuously differentiable, and provide an application of our results to the analysis of quasi-linear demand systems in the presence of measurement errors, in the spirit of Lewbel (2001). Finally, in Section 5, we relate our results to the literature on (quasi-linear) rationalizability with finite

[1]See Amir, Erickson, and Jin (2017) for a treatment of quasi-linear integrability when demand is linear.

[2]Hosoya (2016, Section 3) proves an integrability theorem for demand systems that are defined over an open cone of the positive orthant. While it is possible to apply his results to study quasi-linear integrability, his construction relies on the demand system being continuously differentiable and surjective, an assumption which we do not need to make.

[3]The fact that the maximization of a quasi-linear utility function with a negative definite Hessian matrix delivers a demand system with a symmetric and negative semi-definite substitution matrix is well-known (see, e.g., Vives, 2000, Section 3.1). In this paper, we fully characterize the set of continuous demand systems that are derivable from quasi-linear utility maximization.

or infinite data (Afriat, 1967; Brown and Calsamiglia, 2007; Sákovics, 2013; Dziewulski and Quah, 2014; Nishimura, Ok, and Quah, 2016), and discuss what can be done when demand is not continuous.

2 Definitions and Statement of the Theorem

Notation. Suppose $x, y \in \mathbb{R}^n$ $(n \geq 1)$. We write $x \geq y$ if $x_i \geq y_i$ for every i in $\{1, \ldots, n\}$, $x > y$ if $x \geq y$ and $x \neq y$, and $x >> y$ if $x_i > y_i$ for every i in $\{1, \ldots, n\}$. $\mathbb{R}_+$ (resp. $\mathbb{R}_{++}$) is the set of non-negative (resp. strictly positive) real numbers. $\mathbb{N}$ (resp. $\mathbb{N}^*$) is the set of non-negative (resp. strictly positive) integers. Suppose $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ $(m, n \geq 1)$, and $f : X \to Y$. We say that f is non-decreasing if $f(x) \geq f(x')$ whenever $x \geq x'$. In addition, we denote the range of f by $\mathcal{R}(f)$, the closure of X by $\overline{X}$, the interior of X by $\mathring{X}$, and the boundary of X by ∂X. ∇ denotes the gradient operator, and superscript T denotes the transpose operator. For every $1 \leq i \leq n$, we denote by e_i the i-th vector in the standard basis of $\mathbb{R}^n$, and we let $\mathbf{1}_n = \sum_{j=1}^n e_j$. $\|.\|$ denotes the euclidian norm.

We recall the following definitions:

Definition 1. *Let X be a subset of $\mathbb{R}^n$. We say that X is comprehensive upward if for every $x \in X$, for every $y \in \mathbb{R}^n$, if $y \geq x$, then $y \in X$.*

Definition 2. *Let X be a subset of $\mathbb{R}^n$.*
The comprehensive convex hull of X, denoted $CCH(X)$ is the smallest (by set inclusion) set Y that (a) contains X, (b) is convex, and (c) is comprehensive upward.
The closed comprehensive convex hull of X, denoted $CCCH(X)$ is the smallest closed set Y that satisfies properties (a), (b), and (c).

The existence of $CCH(X)$ and $CCCH(X)$ can be easily established by taking intersections. In addition, $CCH(X) \subseteq CCCH(X)$. We will later establish that $\overline{CCH(X)} = CCCH(X)$ (Lemma 7).

Let $D(.)$ be a function from $\mathbb{R}^n_{++}$ to $\mathbb{R}^n_+$. For every $1 \leq i \leq n$, we denote by $D_i(p)$ the i-th component of vector $D(p)$. $D_i(p)$ is the demand for product i at price vector p. The function $D(.)$ is called a demand system.

We now introduce an outside good (good 0, priced at p_0), and use D to construct a complete demand system $\hat{D}$ as follows: for every $p_0 \in \mathbb{R}_{++}$, $p \in \mathbb{R}^n_{++}$ and $y \in \mathbb{R}_+$,[4]

$$\hat{D}_i(p_0, p, y) = D_i\left(\frac{p}{p_0}\right), \ \forall i \in \{1, \ldots, n\},$$

[4]In some treatments of quasi-linear preferences, q_0 is viewed as "money left over," in which case p_0 should be simply set equal to unity. We view q_0 as an outside good, which represents the rest of the economy, and which can take any strictly positive price p_0. That price can still be normalized to one without loss of generality.

$$\hat{D}_0(p_0, p, y) = \frac{1}{p_0}\left(y - \sum_{i=1}^{n} p_i D_i\left(\frac{p}{p_0}\right)\right).$$

Definition 3. *Demand system $D(\cdot)$ is quasi-linearly integrable if there exist a set $X \subseteq \mathbb{R}^n_+$ and a function $u : X \to \mathbb{R}$ such that for every $(p_0, p, y) \in \mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}_+$ such that $\hat{D}_0(p_0, p, y) \geq 0$, $\hat{D}(p_0, p, y)$ is the unique solution of*

$$\max_{(q_0,q)} (q_0 + u(q)) \ \ s.t.\ \ p_0 q_0 + p \cdot q \leq y,\ \ q_0 \geq 0 \text{ and } q \in X.$$

When this is the case, we say that D can be derived from (X, u). u is called a (direct) utility function for the demand system D.

As mentioned in the introduction, we cannot use Hurwicz and Uzawa (1971) and Jackson (1986)'s classical results to establish integrability of demand system $\hat{D}$, because $\hat{D}_0$ is strictly negative at certain price and income vectors, and condition $\hat{D}_i(\cdot, \cdot, 0) = 0$ is not satisfied.

Next, we define the concept of indirect subutility function:

Definition 4. *We say that D can be indirectly derived from $v : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$ if there exists (X, u) such that D can be derived from (X, u), and $v(p) = u(D(p)) - p \cdot D(p)$ for every $p >> 0$. v is called an indirect subutility function for the demand system D.*

The subutility function we construct for the demand system D is defined over a set X, which may be a proper subset of the non-negative orthant. We will show that that set X is maximal in the following sense:

Definition 5. *We say that (X, u) is maximal for the demand system $D(\cdot)$ if*

- *D can be derived from (X, u), and*
- *For every $(\tilde{X}, \tilde{u})$ such that D can be derived from $(\tilde{X}, \tilde{u})$, the set $\tilde{X} \backslash X$ has Lebesgue measure zero.*

We can now state our quasi-linear integrability theorem:

Theorem 1. *Suppose that the demand system D is continuous. The following are equivalent:*

(i) D is quasi-linearly integrable.

(ii) For every $k \geq 2$ and $(p^i)_{1 \leq i \leq k} \in (\mathbb{R}^n_{++})^k$,

$$(p^2 - p^1) \cdot D(p^1) + (p^3 - p^2) \cdot D(p^2) + \ldots + (p^k - p^{k-1}) \cdot D(p^{k-1}) + (p^1 - p^k) \cdot D(p^k) \geq 0. \quad (1)$$

(iii) D is conservative (i.e., $\int_C D(p) \cdot dp = 0$ for every closed and piecewise-continuously differentiable path C) and satisfies the law of demand (i.e, $(p' - p) \cdot (D(p') - D(p)) \leq 0$ for every $p, p' >> 0$).[5]

[5] A closed and piecewise-continuously differentiable path is a continuous function $p : [0, 1] \longrightarrow \mathbb{R}^n_{++}$ such that p is piecewise $\mathcal{C}^1$ and $p(0) = p(1)$. The line integral $\int_C D(p) \cdot dp$ is defined as $\int_0^1 p'(t) \cdot D(p(t))dt$.

If D is $\mathcal{C}^1$, then the above assertions are equivalent to

(iv) For every $p >> 0$, the substitution matrix $J(p) = \left(\frac{\partial D_i}{\partial p_j}(p)\right)_{1\leq i,j\leq n}$ is symmetric and negative semi-definite.

Moreover, regardless of the differentiability properties of D, if one of the above conditions holds, then there exists (X, u) such that

- *(X, u) is maximal for D,*
- *X is convex and comprehensive upward, and $CCH(\mathcal{R}(D)) \subseteq X \subseteq CCCH(\mathcal{R}(D))$,*
- *u is non-decreasing, concave and upper semi-continuous. In addition, u is continuous on $\mathring{X} \bigcap \mathbb{R}^n_{++}$.*

Finally, if D can be indirectly derived from v, then $\nabla v = -D$, and the function v is unique up to an additive constant.

3 Proof of the Theorem

In this section, we assume that the demand system D is continuous, and prove the theorem in several steps. In Section 3.1, we prove the results related to the indirect subutility function v. Building on these results, we prove that (i) implies (iii) in Section 3.2, and, assuming (iii), we construct a candidate for a direct subutility function $u(\cdot)$ defined over some domain X in Section 3.3. There, we also show that X is convex, comprehensive upward and contains $CCH(\mathcal{R}(D))$, and that u is concave and non-decreasing. Section 3.4 shows that D can indeed be derived from (X, u). Section 3.5 shows that $X \subseteq CCCH(\mathcal{R}(D))$, and that (X, u) is maximal for D. The continuity properties of u are established in Section 3.6. Section 3.7 shows that (ii) is equivalent to (iii). The fact that (iii) is equivalent to (iv) when D is $\mathcal{C}^1$ follows from Lemmas 13 and 14, stated and proven in the appendix.

3.1 On the indirect utility function

Lemma 1. *If D can be indirectly derived from v, then $\nabla v = -D$.*
Moreover, if condition (iii) in Theorem 1 holds, then there exists a function v such that $\nabla v = -D$.
In both cases, the function v is unique up to an additive constant.

Proof. Suppose that D can be indirectly derived from v. There exists (X, u) such that D can be derived from (X, u) and $v(p) = u(D(p)) - p \cdot D(p)$ for every p. Since D can be derived from (X, u), for every $p >> 0$, $D(p)$ solves $\max_{x\in X} u(x) - p \cdot x$. Moreover, for every $p, p' >> 0$, $v(p) \geq u(D(p')) - p \cdot D(p')$.

All we need to do is show that v is differentiable and $\nabla v = -D$. Let $p >> 0$. For every $\varepsilon \in \mathbb{R}^n \backslash \{0\}$ such that $p + \varepsilon >> 0$, let

$$\Delta(\varepsilon) = \frac{v(p+\varepsilon) - v(p) + \varepsilon \cdot D(p)}{\|\varepsilon\|}.$$

Then,

$$\Delta(\varepsilon) \geq \frac{u\left(D(p)\right) - (p+\varepsilon) \cdot D(p) - \left(u\left(D(p)\right) - p \cdot D(p)\right) + \varepsilon \cdot D(p)}{\|\varepsilon\|} = 0.$$

Moreover,

$$\begin{aligned}\Delta(\varepsilon) &\leq \frac{u\left(D(p+\varepsilon)\right) - (p+\varepsilon) \cdot D(p+\varepsilon) - \left(u\left(D(p+\varepsilon)\right) - p \cdot D(p+\varepsilon)\right) + \varepsilon \cdot D(p)}{\|\varepsilon\|}, \\ &= \underbrace{\frac{\varepsilon}{\|\varepsilon\|}}_{\text{bounded}} \cdot \underbrace{\left(D(p) - D(p+\varepsilon)\right)}_{\xrightarrow[\varepsilon \to 0]{} 0, \text{ since } D \text{ is continuous}} \xrightarrow[\varepsilon \to 0]{} 0.\end{aligned}$$

By the sandwich theorem, $\lim_0 \Delta = 0$, v is differentiable, and $\nabla v = -D$.

Next, suppose that condition (iii) holds. Then, by Lemma 13 in the appendix, there exists a function v such that $\nabla v = -D$. The fact that the function v is unique up to an additive constant also follows from Lemma 13. $\square$

3.2 Proof that (i) implies (iii)

Lemma 2. *If D is quasi-linearly integrable, then D is conservative and satisfies the law of demand.*

Proof. Suppose that D can be derived from (X, u), and let $v(p) = u(D(p)) - p \cdot D(p)$ for every p. Then, by Lemma 1, $\nabla v = -D$. Therefore, D has a potential, and, by Lemma 13 in the appendix, D is conservative.

Next, we prove that v is convex. Let $p, p' >> 0$, $\lambda \in [0,1]$, and $p'' = \lambda p + (1-\lambda)p'$. Then,

$$\begin{aligned} v\left(p''\right) &= \lambda\left(u(D(p'')) - p \cdot D(p'')\right) + (1-\lambda)\left(u(D(p'')) - p' \cdot D(p'')\right), \\ &\leq \lambda\left(u(D(p)) - p \cdot D(p)\right) + (1-\lambda)\left(u(D(p')) - p' \cdot D(p')\right), \\ &= \lambda v(p) + (1-\lambda) v(p'). \end{aligned}$$

It follows that v is convex. Therefore, by Lemma 14 in the appendix, D satisfies the law of demand. $\square$

3.3 Construction of (X, u) and basic properties

In the following, we assume that (iii) holds. By Lemma 1, there exists a function v such that $\nabla v = -D$. We guess that

$$V(p_0, p, y) = \frac{y}{p_0} + v\left(\frac{p}{p_0}\right)$$

is an indirect utility function for the demand system $\hat{D}$. If so, then the corresponding expenditure function is given by

$$E(p_0, p, \bar{u}) = p_0\left(\bar{u} - v\left(\frac{p}{p_0}\right)\right),$$

where $\bar{u}$ is a target utility level. We use Jackson (1986) and Krishna and Sonnenschein (1990)'s duality formulas to construct a direct utility function:

$$\begin{aligned}
U(q_0, q) &= \sup\left\{\bar{u} \in \mathbb{R} : \ p_0 q_0 + \tilde{p} \cdot q \geq p_0\left(\bar{u} - v\left(\frac{\tilde{p}}{p_0}\right)\right), \ \forall p_0 > 0, \ \forall \tilde{p} >> 0\right\}, \\
&= \sup\left\{\bar{u} \in \mathbb{R} : \ q_0 + p \cdot q \geq \bar{u} - v\left(p\right), \ \forall p >> 0\right\}, \\
&= \sup\left\{\bar{u} \in \mathbb{R} : \ \bar{u} - q_0 \leq p \cdot q + v\left(p\right), \ \forall p >> 0\right\}, \\
&= q_0 + \sup\left\{\bar{v} \in \mathbb{R} : \ \bar{v} \leq p \cdot q + v\left(p\right), \ \forall p >> 0\right\}, \\
&= q_0 + \inf_{p>>0}\left\{p \cdot q + v(p)\right\}.
\end{aligned}$$

This gives us a candidate for $u(.)$:

$$u(q) = \inf_{p>>0}\left\{p \cdot q + v(p)\right\},$$

and a candidate for the set X:

$$X = \left\{q \in \mathbb{R}^n_+ : \ u(q) \text{ is finite}\right\}.$$

Let $\phi_q(p) = p \cdot q + v(p)$ for every $q \geq 0$ and $p >> 0$. Then, $u(q) = \inf_{p>>0} \phi_q(p)$. We first establish some basic properties of the pair (X, u):

Lemma 3. *The set X has the following properties:*

(a) X contains $\mathcal{R}(D)$.

(b) X is convex.

(c) X is comprehensive upward.

Therefore, $CCH\left(\mathcal{R}(D)\right) \subseteq X$.

Lemma 4. *The function u has the following properties:*

(a) For every $p >> 0$, $u\left(D(p)\right) = p \cdot D(p) + v(p)$.

(b) u is concave.

(c) u is non-decreasing.

Lemmas 3 and 4 are proven jointly:

Proof. Since $\phi_q\left(\mathbf{1}_n\right) \in \mathbb{R}$, $u(q) < \infty$. Therefore, $u(q)$ is finite if and only if $u(q) > -\infty$. Moreover, since D satisfies the law of demand, it follows from Lemma 14 in the appendix that v is convex. Therefore, $\phi_q(\cdot)$ is convex for every $q \geq 0$.

Let $q \in \mathcal{R}(D)$. There exists $\hat{p} >> 0$ such that $q = D(\hat{p}) = -\nabla v(\hat{p})$. Therefore, $\nabla \phi_q(\hat{p}) = 0$ and, by convexity, $\hat{p}$ is a global minimizer of the function $\phi_q(.)$. It follows that $u(q) = \phi_q(\hat{p})$ is finite, and that $q \in X$. This establishes part (a) in Lemmas 3 and 4.

Let $(q, q') \in X^2$ and $\lambda \in [0, 1]$. Let $q'' = \lambda q + (1-\lambda)q'$. Then, for every $p >> 0$,

$$\begin{aligned}
\phi_{q''}(p) &= p \cdot (\lambda q + (1-\lambda)q') + v(p), \\
&= \lambda\left(p \cdot q + v(p)\right) + (1-\lambda)\left(p \cdot q' + v(p)\right), \\
&= \lambda \phi_q(p) + (1-\lambda)\phi_{q'}(p), \\
&\geq \lambda u(q) + (1-\lambda)u(q') > -\infty.
\end{aligned}$$

It follows that $u(q'')$ is finite, $q'' \in X$, and $u(q'') \geq \lambda u(q) + (1-\lambda)u(q')$. This establishes part (b) in Lemmas 3 and 4.

Next assume that $q \in X$ and $q' \geq q$. Then, for every $p >> 0$,

$$\phi_{q'}(p) = p \cdot q' + v(p) \geq \phi_q(p) \geq u(q) > -\infty.$$

Therefore, $u(q')$ is finite, $q' \in X$ and $u(q') \geq u(q)$. This establishes part (c) in Lemmas 3 and 4.

Finally, since X is a convex and comprehensive upward set that contains $\mathcal{R}(D)$, it follows that $CCH\left(\mathcal{R}(D)\right) \subseteq X$. □

3.4 Proof that D can be derived from (X, u)

In this section, we continue to assume that condition (iii) holds, and we show that D can indeed be derived from the pair (X, u) constructed in Section 3.3.

Lemma 5. *For every (p_0, p, y) such that $\hat{D}_0(p_0, p, y) \geq 0$, for every $(q_0, q) \in \mathbb{R}_+ \times X$ such that $q \neq D\left(\frac{p}{p_0}\right)$ and $p_0 q_0 + p \cdot q \leq y$,*

$$q_0 + u\left(q\right) < \hat{D}_0\left(p_0, p, y\right) + u\left(D\left(\frac{p}{p_0}\right)\right).$$

Proof. Let $U(q_0,q)=q_0+u(q)$ for every $(q_0,q)\in\mathbb{R}_+\times X$, and let $(p_0,p,y)\in\mathbb{R}_{++}\times\mathbb{R}^n_{++}\times\mathbb{R}_+$. Since U is strictly increasing in q_0, it is enough to prove that

$$U(q_0,q)<U\left(\hat{D}_0(p_0,p,y),D\left(\frac{p}{p_0}\right)\right),$$

for every $(q_0,q)\in\mathbb{R}\times\left(X\backslash\left\{D\left(\frac{p}{p_0}\right)\right\}\right)$ such that $p_0q_0+p\cdot q=y$.

Let $q\in X$ and $q_0=\frac{1}{p_0}(y-p\cdot q)$. Then,

$$\begin{aligned}
U(q_0,q)&=\frac{1}{p_0}(y-p\cdot q)+\inf_{\tilde{p}>>0}\{\tilde{p}\cdot q+v(\tilde{p})\},\\
&\le\frac{1}{p_0}(y-p\cdot q)+\frac{p}{p_0}\cdot q+v\left(\frac{p}{p_0}\right),\ \text{by definition of the inf},\\
&=\frac{y}{p_0}+v\left(\frac{p}{p_0}\right),\\
&=\frac{1}{p_0}\left(y-p\cdot D\left(\frac{p}{p_0}\right)\right)+\frac{p}{p_0}\cdot D\left(\frac{p}{p_0}\right)+v\left(\frac{p}{p_0}\right),\\
&=\hat{D}_0(p_0,p,y)+u\left(D\left(\frac{p}{p_0}\right)\right),\ \text{by Lemma 4-(a)},\\
&=U\left(\hat{D}_0(p_0,p,y),D\left(\frac{p}{p_0}\right)\right).
\end{aligned}$$

Next, assume that

$$U(q_0,q)=U\left(\hat{D}_0(p_0,p,y),D\left(\frac{p}{p_0}\right)\right).$$

Then,

$$\frac{y}{p_0}-\frac{p}{p_0}\cdot q+\inf_{\tilde{p}>>0}\{\tilde{p}\cdot q+v(\tilde{p})\}=\frac{y}{p_0}+v\left(\frac{p}{p_0}\right),$$

i.e., $\inf_{\tilde{p}>>0}\phi_q(\tilde{p})=\phi_q\left(\frac{p}{p_0}\right)$. Therefore, $\frac{p}{p_0}$ minimizes $\phi_q(.)$, and $\nabla\,\phi_q(\tilde{p})|_{\tilde{p}=\frac{p}{p_0}}=0$. It follows that

$$q=-\nabla\,v(\tilde{p})|_{\tilde{p}=\frac{p}{p_0}}=D\left(\frac{p}{p_0}\right).$$

This concluses the proof. □

3.5 Maximality of (X,u)

In this section, we continue to assume that condition (iii) holds, and prove that the pair (X,u) constructed in Section 3.3 is indeed maximal.

Lemma 6. *Suppose that D can be derived from $(\tilde{X},\tilde{u})$. Then, $\tilde{X}\subseteq CCCH(\mathcal{R}(D))$.*

Proof. If $CCCH(\mathcal{R}(D))=\mathbb{R}^n_+$, then there is nothing to prove, so suppose that $CCCH(\mathcal{R}(D))\subsetneq\mathbb{R}^n_+$, let $x\in\mathbb{R}^n_+\backslash CCCH(\mathcal{R}(D))$, and assume for a contradiction that $x\in\tilde{X}$. $\{x\}$ and

$CCCH\left(\mathcal{R}(D)\right)$ are both convex, $\{x\} \bigcap CCCH\left(\mathcal{R}(D)\right) = \emptyset$, $\{x\}$ is compact, and $CCCH\left(\mathcal{R}(D)\right)$ is closed. By the separating hyperplane theorem, there exist $\tilde{p} \in \mathbb{R}^n$ and $(c_1, c_2) \in \mathbb{R}^2$ such that for every $q \in CCCH\left(\mathcal{R}(D)\right)$,

$$\tilde{p} \cdot x < c_1 < c_2 < \tilde{p} \cdot q.$$

Assume for a contradiction that $\tilde{p}_i < 0$ for some $i \in \{1, \ldots, n\}$. Let $q \in CCCH\left(\mathcal{R}(D)\right)$, and, for every $\mu \geq 0$, let $r(\mu) = q + \mu e_i$. Since $CCCH\left(\mathcal{R}(D)\right)$ is comprehensive upward, $r(\mu) \in CCCH\left(\mathcal{R}(D)\right)$ for every $\mu \geq 0$. Since $\tilde{p} \cdot r(\mu) = \tilde{p} \cdot q + \mu \tilde{p}_i \underset{\mu \to \infty}{\longrightarrow} -\infty$, there exists $\mu \geq 0$ such that $\tilde{p} \cdot r(\mu) \leq \tilde{p} \cdot x$, which is a contradiction. Therefore, $\tilde{p} \geq 0$.

Next, choose $\varepsilon > 0$ such that $\tilde{p} \cdot x + \varepsilon \sum_{i=1}^{n} x_i < c_1$, and let $p = \tilde{p} + \varepsilon \mathbf{1}_n$. Then, $p >> 0$, and for every $q \in CCCH\left(\mathcal{R}(D)\right)$,

$$p \cdot x < c_1 < c_2 < \tilde{p} \cdot q \leq \tilde{p} \cdot q + \varepsilon \sum_{i=1}^{n} q_i = p \cdot q.$$

It follows that, for every $\lambda > 0$ and $q \in CCCH\left(\mathcal{R}(D)\right)$,

$$(\lambda p) \cdot x < \lambda c_1 < \lambda c_2 < (\lambda p) \cdot q.$$

Therefore, for every $\lambda > 0$ and $q \in CCCH\left(\mathcal{R}(D)\right)$, $\lambda p \cdot x < \lambda(c_1 - c_2) + \lambda p \cdot q$. In particular, for every $\lambda > 0$,

$$\lambda p \cdot x < \lambda(c_1 - c_2) + \lambda p \cdot D(\lambda p).$$

In addition, since D can be derived from $(\tilde{X}, \tilde{u})$, we have that

$$\tilde{u}(x) - \lambda p \cdot x \leq \tilde{u}\left(D(\lambda p)\right) - \lambda p \cdot D(\lambda p), \ \forall \lambda > 0.$$

It follows that, for every λ,

$$\begin{aligned} \tilde{u}(x) &\leq \tilde{u}\left(D(\lambda p)\right) + \lambda p \cdot x - \lambda p \cdot D(\lambda p), \\ &< \tilde{u}\left(D(\lambda p)\right) + \lambda \underbrace{(c_1 - c_2)}_{<0}. \end{aligned}$$

All we need to do now is show that $\tilde{u}\left(D(\lambda p)\right)$ is non-increasing in λ. Once this is established, it follows immediately that the right-hand side of the above inequality goes to $-\infty$ as λ goes to $+\infty$. Therefore, $u(x) = -\infty$, and $x \notin \tilde{X}$, a contradiction.

Let $0 < \lambda < \lambda'$. Since D can be derived from $(\tilde{X}, \tilde{u})$,

$$\tilde{u}\left(D(\lambda p)\right) - \lambda p \cdot D(\lambda p) \geq \tilde{u}\left(D(\lambda' p)\right) - \lambda p \cdot D(\lambda' p).$$

Therefore,

$$
\begin{aligned}
\tilde{u}\left(D(\lambda p)\right) - \tilde{u}\left(D(\lambda' p)\right) &\geq \lambda p \cdot \left(D(\lambda p) - D(\lambda' p)\right), \\
&= \frac{\lambda}{\lambda' - \lambda}(\lambda' p - \lambda p) \cdot \left(D(\lambda p) - D(\lambda' p)\right), \\
&\geq 0,
\end{aligned}
$$

since D satisfies the law of demand. This concludes the proof. $\square$

It follows immediately from Lemma 6 that $X \subseteq CCCH\left(\mathcal{R}(D)\right)$. The following technical lemma will allow us to prove that (X, u) is maximal for D:

Lemma 7. *If Y is a subset of $\mathbb{R}^n$, then $CCCH(Y) = \overline{CCH(Y)}$.*

Proof. We first prove that $\overline{CCH(Y)} \subseteq CCCH(Y)$. Let $y \in \overline{CCH(Y)}$. There exists a sequence $(y^k)_{k\geq 0}$ in $CCH(Y)$ such that $y^k \underset{k\to\infty}{\longrightarrow} y$. Since $CCH(Y) \subseteq CCCH(Y)$, it follows that $y^k \in CCCH(Y)$ for every $k \geq 0$. Since $CCCH(Y)$ is closed, we can conclude that $y = \lim_{k\to\infty} y^k \in CCCH(Y)$.

Next, we show that $CCCH(Y) \subseteq \overline{CCH(Y)}$. Towards this goal, we show that $\overline{CCH(Y)}$ is convex and comprehensive upward.

Let us start with convexity. Let $y, y' \in \overline{CCH(Y)}$ and $\lambda \in [0, 1]$. Let $y'' = \lambda y + (1-\lambda)y'$. There exist two sequences $(y^k)_{k\geq 0}$ and $(y'^k)_{k\geq 0}$ in $CCH(Y)$ such that $y^k \underset{k\to\infty}{\longrightarrow} y$ and $y'^k \underset{k\to\infty}{\longrightarrow} y'$. For every $k \geq 0$, let $y''^k = \lambda y^k + (1-\lambda)y'^k$. Then, since $CCH(Y)$ is convex, $y''^k \in CCH(Y)$ for every k. Therefore, $y'' = \lim_{k\to\infty} y''^k \in \overline{CCH(Y)}$, and $\overline{CCH(Y)}$ is convex.

Next, we turn our attention to comprehensiveness. Let $y \in \overline{CCH(Y)}$ and $y' \geq y$. There exists a sequence $(y^k)_{k\geq 0}$ in $CCH(Y)$ such that $y^k \underset{k\to\infty}{\longrightarrow} y$. For every $k \geq 0$, let $y'^k = y^k + y' - y$. Then, for every $k \geq 0$, $y'^k \geq y^k$, and, by comprehensiveness, $y'^k \in CCH(Y)$. Therefore, $y' = \lim_{k\to\infty} y'^k \in \overline{CCH(Y)}$, and $\overline{CCH(Y)}$ is comprehensive upward.

We can conclude: $\overline{CCH(Y)}$ is closed, convex, comprehensive upward, and contains Y. Therefore, $CCCH(Y) \subseteq \overline{CCH(Y)}$. $\square$

Lemma 8. *(X, u) is maximal for D.*

Proof. Suppose that D can be derived from $(\tilde{X}, \tilde{u})$. Then, by Lemmas 6 and 7, $\tilde{X} \subseteq \overline{CCH\left(\mathcal{R}(D)\right)}$. Moreover, by Lemma 3, $CCH\left(\mathcal{R}(D)\right) \subseteq X$. Therefore,

$$
\begin{aligned}
\tilde{X} \backslash X &\subseteq \overline{CCH\left(\mathcal{R}(D)\right)} \backslash CCH\left(\mathcal{R}(D)\right), \\
&\subseteq \overline{CCH\left(\mathcal{R}(D)\right)} \backslash \overset{\circ}{CCH}\left(\mathcal{R}(D)\right), \\
&= \partial CCH\left(\mathcal{R}(D)\right).
\end{aligned}
$$

Since $CCH\left(\mathcal{R}(D)\right)$ is a convex subset of $\mathbb{R}^n$, its boundary $\partial CCH\left(\mathcal{R}(D)\right)$ has Lebesgue measure zero (see, e.g., Lemma 2.24 in Dudley (1999)). Therefore, $\tilde{X} \backslash X$ has measure zero, and (X, u) is maximal for D. $\square$

3.6 (Upper semi-)continuity of u

In this section, we continue to assume that condition (iii) holds, and establish the continuity properties of the utility function u constructed in Section 3.3.

Lemma 9. *u is upper semi-continuous.*

Proof. Let $x_0 \in X$ and $\varepsilon > 0$. By definition of u, there exists $p >> 0$ such that

$$p \cdot x_0 + v(p) < u(x_0) + \frac{\varepsilon}{2}.$$

In addition, by continuity of the inner product, there exists $\eta > 0$ such that $p \cdot x < p \cdot x_0 + \frac{\varepsilon}{2}$ for every $x \in \mathbb{R}^n_+$ such that $\|x - x_0\| < \eta$. Therefore, for every $x \in X$ such that $\|x - x_0\| < \eta$,

$$u(x) \leq p \cdot x + v(p) < p \cdot x_0 + \frac{\varepsilon}{2} + v(p) < u(x_0) + \varepsilon.$$

Therefore, u is upper semi-continuous at point x_0. □

Lemma 10. *u is continuous on $\mathring{X} \bigcap \mathbb{R}^n_{++}$.*

Proof. Let $x \in \mathring{X} \bigcap \mathbb{R}^n_{++}$. Assume for a contradiction that u is not lower semi-continuous at x. There exist an $\varepsilon_0 > 0$ and a sequence $(x^k)_{k \geq 0}$ in X such that $x^k \underset{k \to \infty}{\longrightarrow} x$ and $u(x^k) < u(x) - \varepsilon_0$ for every k. By definition of u, for every $k \geq 0$, there exists $p^k \in \mathbb{R}^n_{++}$ such that

$$p^k \cdot x^k + v(p^k) < u(x) - \varepsilon_0.$$

Since $x \in \mathring{X} \bigcap \mathbb{R}^n_{++}$, there exists $\kappa > 0$ such that $x - 2\kappa \mathbf{1}_n \in X$. Since $x^k \underset{k \to \infty}{\longrightarrow} x$, $x^k >> x - \kappa \mathbf{1}_n$ for high enough k. Assume for a contradiction that $(p^k)_{k \geq 0}$ is not bounded above. Assume without loss of generality that $(p^k_1)_{k \geq 0}$ is not bounded above, and extract a subsequence $\left(p^{\xi(k)}\right)_{k \geq 0}$ such that $p^{\xi(k)}_1 \underset{k \to \infty}{\longrightarrow} \infty$. Then, for high enough k,

$$\begin{aligned}
p^{\xi(k)} \cdot x^{\xi(k)} + v\left(p^{\xi(k)}\right) &> p^{\xi(k)} \cdot (x - \kappa \mathbf{1}_n) + v\left(p^{\xi(k)}\right), \\
&= \kappa \mathbf{1}_n \cdot p^{\xi(k)} + p^{\xi(k)} \cdot (x - 2\kappa \mathbf{1}_n) + v\left(p^{\xi(k)}\right), \\
&\geq \kappa \sum_{i=1}^{n} p^{\xi(k)}_i + u\left(x - 2\kappa \mathbf{1}_n\right), \\
&\geq \kappa p^{\xi(k)}_1 + u\left(x - 2\kappa \mathbf{1}_n\right),
\end{aligned}$$

which goes to infinity as k goes to infinity. It follows that $p^{\xi(k)} \cdot x^{\xi(k)} + v\left(p^{\xi(k)}\right) \underset{k \to \infty}{\longrightarrow} \infty$, which contradicts the fact that $p^{\xi(k)} \cdot x^{\xi(k)} + v\left(p^{\xi(k)}\right) < u(x) - \varepsilon_0$ for every k.

Therefore, $\left(p^k\right)_{k \geq 0}$ is bounded. Let $M > 0$ such that $p^k_i \leq M$ for every $1 \leq i \leq n$ and $k \geq 0$. Since $\nabla v = -D \leq 0$, v is non-increasing, and $v^k \equiv v(p^k) \geq v\left(M \mathbf{1}_n\right)$ for every k. Therefore, sequence $(v^k)_{k \geq 0}$ is bounded below. In addition, $v^k \leq u(x) - \varepsilon_0$ for every k, so

$(v^k)_{k\geq 0}$ is also bounded above. We can conclude: since $\left(p^k\right)_{k\geq 0}$ and $(v^k)_{k\geq 0}$ are bounded, there exist $\psi : \mathbb{N}^* \to \mathbb{N}^*$ strictly increasing, $v \in \mathbb{R}$ and $p \in \mathbb{R}^n_+$ such that $p^{\psi(k)} \underset{k\to\infty}{\longrightarrow} p$ and $v^{\psi(k)} \underset{k\to\infty}{\longrightarrow} v$. Taking limits in inequality $p^{\psi(k)} \cdot x^{\psi(k)} + v^{\psi(k)} < u(x) - \varepsilon_0$, it follows that

$$p \cdot x + v \leq u(x) - \varepsilon_0.$$

Yet, for every $k \geq 0$, $p^{\psi(k)} \cdot x + v\left(p^{\psi(k)}\right) \geq u(x)$. Taking limits again, it follows that

$$u(x) \leq p \cdot x + v,$$

which is a contradiction. □

3.7 Proof that (ii) is equivalent to (iii)

Lemma 11. *Let D be a continuous demand system. Then, D is conservative and satisfies the law of demand if and only if condition* (1) *holds for every $k \geq 2$ and $(p^i)_{1\leq i\leq k} \in (\mathbb{R}^n_{++})^k$.*

Proof. Assume that D is conservative and satisfies the law of demand. Then, by Lemma 5, there exists (X, u) such that D can be derived from (X, u). Assume for a contradiction that

$$(p^2 - p^1) \cdot D(p^1) + (p^3 - p^2) \cdot D(p^2) + \ldots + (p^k - p^{k-1}) \cdot D(p^{k-1}) + (p^1 - p^k) \cdot D(p^k) < 0$$

for some $k \geq 2$ and $(p^i)_{1\leq i\leq k} \in (\mathbb{R}^n_{++})^k$. For every $x > 0$ and $1 \leq i \leq k$, define

$$y^i(x) = x + \sum_{j=1}^{i-1} (p^{j+1} - p^j) \cdot D(p^j).$$

For x high enough, $p^i \cdot D(p^i) \leq y^i(x)$ for every i. Fix such an x. We drop x from the notation in the following. By definition of $(y^i)_{1\leq i\leq k}$, we have that, for every $1 \leq i \leq k-1$,

$$p^{i+1} \cdot D(p^i) + y^i - p^i \cdot D(p^i) = y^{i+1}.$$

Therefore, by revealed preference,

$$y^i - p^i \cdot D(p^i) + u(D(p^i)) \leq y^{i+1} - p^{i+1} \cdot D(p^{i+1}) + u(D(p^{i+1})).$$

This implies in particular that

$$y^1 - p^1 \cdot D(p^1) + u(D(p^1)) \leq y^k - p^k \cdot D(p^k) + u(D(p^k)). \tag{2}$$

Moreover,

$$y^k - p^k \cdot D(p^k) + p^1 \cdot D(p^k) = x + \sum_{j=1}^{k-1}(p^{j+1} - p^j) \cdot D(p^j) + (p^1 - p^k) \cdot D(p^k) < x = y^1.$$

By revealed preference,

$$y^k - p^k \cdot D(p^k) + u(D(p^k)) < y^1 - p^1 \cdot D(p^1) + u(D(p^1)),$$

which contradicts condition (2). Therefore, condition (1) holds for every $k \geq 2$ and $(p^i)_{1\leq i\leq k} \in (\mathbb{R}^n_{++})^k$.

Conversely, assume that condition (1) holds. Applying that condition with $k = 2$ implies immediately that D satisfies the law of demand. Let $p(\cdot)$ be a $\mathcal{C}^1$ path such that $p(0) = p(1)$. (The proof in the case where p is piecewise-$\mathcal{C}^1$ is analogous, but involves more tedious notation.[6]) We define the following sequence of functions: For every $k \geq 1$ and $t \in [0, 1)$,

$$\phi^k(t) = k\left(p\left(\frac{\lfloor kt\rfloor + 1}{k}\right) - p\left(\frac{\lfloor kt\rfloor}{k}\right)\right) \cdot D\left(p\left(\frac{\lfloor kt\rfloor}{k}\right)\right),$$

where $\lfloor\cdot\rfloor$ is the floor function.[7] We first show that $(\phi^k)_{k\geq1}$ converges pointwise to $t \in [0, 1) \mapsto p'(t) \cdot D(p(t))$. Clearly, $\lfloor kt\rfloor/k \underset{k\to\infty}{\longrightarrow} t$ for every t. Therefore, by continuity of p and D, $D(p(\lfloor kt\rfloor/k))$ converges pointwise to $D(p(t))$ as $k \to \infty$. Next, we turn our attention to the term $k\left(p\left(\frac{\lfloor kt\rfloor+1}{k}\right) - p\left(\frac{\lfloor kt\rfloor}{k}\right)\right)$. By Taylor's theorem, for every $1 \leq i \leq n$, there exist functions g_i and h_i such that $\lim_0 g_i = \lim_0 h_i = 0$, and for every k,

$$\begin{aligned} p_i\left(\frac{\lfloor kt\rfloor + 1}{k}\right) &= p_i(t) + \left(\frac{\lfloor kt\rfloor + 1}{k} - t\right)p_i'(t) + \left(\frac{\lfloor kt\rfloor + 1}{k} - t\right)g_i\left(\frac{\lfloor kt\rfloor + 1}{k} - t\right), \\ p_i\left(\frac{\lfloor kt\rfloor}{k}\right) &= p_i(t) + \left(\frac{\lfloor kt\rfloor}{k} - t\right)p_i'(t) + \left(\frac{\lfloor kt\rfloor}{k} - t\right)h_i\left(\frac{\lfloor kt\rfloor}{k} - t\right). \end{aligned}$$

Subtracting, and multiplying by k, we obtain:

$$k\left(p_i\left(\frac{\lfloor kt\rfloor + 1}{k}\right) - p_i\left(\frac{\lfloor kt\rfloor}{k}\right)\right) = p_i'(t) + (\lfloor kt\rfloor + 1 - tk)\, g_i\left(\frac{\lfloor kt\rfloor + 1}{k} - t\right)$$

[6] If p is $\mathcal{C}^1$ on $[a^0, a^1]$, $[a^1, a^2]$, …, and $[a^{m-1}, a^m]$, where $a^0 = 0$ and $a^m = 1$, then it can be shown using the same method that the line integral $\int_C D(p) \cdot dp$ is equal to

$$\lim_{k\to\infty}\sum_{j=0}^{m-1}\sum_{i=1}^{k-1}\left(p\left(a^j + \frac{i+1}{k}(a^{j+1} - a^j)\right) - p\left(a^j + \frac{i}{k}(a^{j+1} - a^j)\right)\right) \cdot D\left(p\left(a^j + \frac{i}{k}(a^{j+1} - a^j)\right)\right),$$

which is non-negative by condition (1).

[7] Recall that $\lfloor x\rfloor$ is the largest integer not exceeding x.

$$- \left(\lfloor kt \rfloor - tk\right) h_i \left(\frac{\lfloor kt \rfloor}{k} - t\right).$$

Since $\lfloor kt \rfloor + 1 - tk$ and $\lfloor kt \rfloor - tk$ are bounded and $\lim_0 g_i = \lim_0 h_i = 0$, the above expression converges to $p_i'(t)$ as $k \to \infty$. Therefore, $(\phi^k)_{k\geq 1}$ converges pointwise to $t \in [0,1) \mapsto p'(t) \cdot D(p(t))$.

Next, we show that the sequence (ϕ^k) is uniformly bounded by an integrable function. Using the Cauchy-Schwarz inequality, we obtain that, for every k and t,

$$\begin{aligned}
|\phi^k(t)| &\leq k \left\| p\left(\frac{\lfloor kt \rfloor + 1}{k}\right) - p\left(\frac{\lfloor kt \rfloor}{k}\right)\right\| \left\| D\left(p\left(\frac{\lfloor kt \rfloor}{k}\right)\right)\right\|, \\
&\leq \sqrt{n} \max_{1 \leq i \leq n} \max_{0 \leq x \leq 1} |p_i'(x)| \max_{x \in [0,1]} \|D(p(x))\| \equiv M,
\end{aligned}$$

where the second inequality follows by the mean value theorem. By continuity and compactness, all of the above maxima exist, and M is therefore finite. Since $t \in [0,1) \mapsto M$ is obviously integrable, we can apply the dominated convergence theorem to obtain that

$$\lim_{k\to\infty} \int_{[0,1)} \phi^k(t)dt = \int_{[0,1)} p'(t) \cdot D(p(t))dt = \int_0^1 p'(t) \cdot D(p(t))dt.$$

Note that, for every $k \geq 1$,

$$\int_{[0,1)} \phi^k(t)dt = \sum_{j=1}^{k-1} \left(p\left(\frac{j+1}{k}\right) - p\left(\frac{j}{k}\right)\right) \cdot D\left(p\left(\frac{j}{k}\right)\right) \geq 0,$$

where the inequality follows from condition (1) (recall that, by assumption, $p(0) = p(1)$). Therefore, $\int_0^1 p'(t) \cdot D(p(t))dt \geq 0$.

Next, let $\tilde{p}(t) = p(1-t)$ for every $t \in [0,1]$. Then, $\tilde{p}$ is $\mathcal{C}^1$, and $\tilde{p}(0) = \tilde{p}(1) = 0$. Therefore, $\int_0^1 \tilde{p}'(t) \cdot D(\tilde{p}(t))dt \geq 0$. Moreover,

$$\begin{aligned}
\int_0^1 \tilde{p}'(t) \cdot D(\tilde{p}(t))dt &= \int_0^1 -p'(1-t) \cdot D(p(1-t))dt, \\
&\underset{x=1-t}{=} \int_1^0 p'(x) \cdot D(p(x))dx, \\
&= -\int_0^1 p'(x) \cdot D(p(x))dx \leq 0.
\end{aligned}$$

It follows that $\int_0^1 p'(t) \cdot D(p(t))dt = 0$. Therefore, D is conservative. $\square$

4 Additional Results

In this section, we fix D, a continuous and quasi-linearly integrable demand system, and derive additional results on the pair (X, u) constructed in Section 3.3. We also provide an application to the analysis of quasi-linear demand systems in the presence of measurement errors.

4.1 More on X

Proposition 1. *$X = \mathbb{R}^n_+$ if and only if $p \in \mathbb{R}_{++} \mapsto \sum_{i=1}^n D_i(p\mathbf{1}_n)$ is integrable on $[1, \infty)$.*

Proof. We know from Lemma 3-(c) that X is comprehensive upward. It follows that $X = \mathbb{R}^n_+$ if and only if $0 \in X$. This holds if and only if $\inf_{\tilde{p}>>0} \phi_0(\tilde{p}) = \inf_{\tilde{p}>>0} v(\tilde{p}) > -\infty$. We claim that

$$\inf_{\tilde{p}>>0} v(\tilde{p}) = \inf_{p\in\mathbb{R}_{++}} v(p\mathbf{1}_n).$$

Since $\mathbb{R}_{++}\mathbf{1}_n \subseteq \mathbb{R}^n_{++}$, it is clear that $\inf_{\tilde{p}>>0} v(\tilde{p}) \leq \inf_{p\in\mathbb{R}_{++}} v(p\mathbf{1}_n)$. Next, let $\tilde{p} \in \mathbb{R}^n_{++}$, and $\hat{p} \equiv \max_{1\leq i\leq n} \tilde{p}_i$. Since $\nabla v = -D \leq 0$, v is non-increasing. Therefore,

$$v(\tilde{p}) \geq v(\hat{p}\mathbf{1}_n) \geq \inf_{p\in\mathbb{R}_{++}} v(p\mathbf{1}_n).$$

Since the above inequality holds for every $\tilde{p} >> 0$, it follows that $\inf_{\tilde{p}>>0} v(\tilde{p}) \geq \inf_{p\in\mathbb{R}_{++}} v(p\mathbf{1}_n)$, which establishes the claim.

Since $\frac{d}{dp} v(p\mathbf{1}_n) = -\sum_{i=1}^n D_i(p\mathbf{1}_n)$, the function $p \mapsto v(p\mathbf{1}_n)$ is non-increasing. It follows that

$$\inf_{p\in\mathbb{R}_{++}} v(p\mathbf{1}_n) = \lim_{p\to\infty} v(p\mathbf{1}_n).$$

Therefore, $X = \mathbb{R}^n_+$ if and only if $\lim_{p\to\infty} v(p\mathbf{1}_n) > -\infty$.

By Lemma 13 in the appendix, there exists $\alpha \in \mathbb{R}$ such that for every $p > 0$,

$$\begin{aligned} v(p\mathbf{1}_n) &= \alpha - \int_0^1 \sum_{i=1}^n (p-1) D_i\left((1-t+tp)\mathbf{1}_n\right) dt, \\ &= \alpha - \int_1^p \sum_{i=1}^n D_i\left(r\mathbf{1}_n\right) dr, \end{aligned}$$

where the second line follows by the change of variable $r = (1 - t + tp)$. Therefore,

$$\lim_{p\to\infty} v(p\mathbf{1}_n) = \alpha - \int_1^{+\infty} \sum_{i=1}^n D_i\left(r\mathbf{1}_n\right) dr,$$

which, since demand is non-negative, is finite if and only if $p \in \mathbb{R}_{++} \mapsto \sum_{i=1}^n D_i(p\mathbf{1}_n)$ is integrable on $[1, \infty)$. □

It is interesting to notice that the sub-utility function is defined everywhere if and only if consumer surplus (defined as the surplus the consumer receives from the mere existence of the market) is finite (and thus well-defined). Note that consumer surplus is not finite with logit or CES demands without an outside option. Of course, no matter whether consumer surplus is finite, the compensating variation and the equivalent variation when moving from p to p' are both equal to $v(p) - v(p')$ (as long as the non-negativity constraint on the outside good is not binding).

Notice also that, since demand is non-negative, $p \in \mathbb{R}_{++} \mapsto \sum_{i=1}^{n} D_i(p\mathbf{1}_n)$ is integrable on $[1, \infty)$ if and only if $p \in \mathbb{R}_{++} \mapsto D_i(p\mathbf{1}_n)$ is integrable on $[1, \infty)$ for every i. The following result follows immediately from Proposition 1:

Corollary 1. *If $X = \mathbb{R}_+^n$, then $\lim_{p\to\infty} \sum_{i=1}^{n} D_i(p\mathbf{1}_n) = 0$.*

Proof. Let $f(p) = \sum_{i=1}^{n} D_i(p\mathbf{1}_n)$ for every $p \in \mathbb{R}_{++}$. Note that, for every $0 < p < p'$,

$$f(p') - f(p) = \frac{1}{p' - p}(p'\mathbf{1}_n - p\mathbf{1}_n) \cdot (D(p'\mathbf{1}_n) - D(p\mathbf{1}_n)) \leq 0,$$

since D satisfies the law of demand. Therefore, f is non-increasing, $\lim_\infty f$ exists, and $\inf_{\mathbb{R}_{++}} f = \lim_\infty f$.

Suppose $\lim_\infty f = l > 0$. Then, $\int_1^\infty f \geq \int_1^\infty l = +\infty$. Therefore, f is not integrable, and, by Proposition 1, $X \neq \mathbb{R}_+^n$. □

Again, since demand is non-negative, $\lim_{p\to\infty} \sum_{i=1}^{n} D_i(p\mathbf{1}_n) = 0$ if and only if $D_i(p\mathbf{1}_n) \underset{p\to\infty}{\longrightarrow} 0$ for every i. Next, we prove the (almost) converse of Corollary 1:

Proposition 2. *If $\lim_{p\to\infty} \sum_{i=1}^{n} D_i(p\mathbf{1}_n) = 0$, then $\mathbb{R}_{++}^n \subseteq X$.*

Proof. Let $q >> 0$. Let $\underline{q} = \min_{1\leq i\leq n} q_i$. Suppose $\lim_{p\to\infty} \sum_{i=1}^{n} D_i(p\mathbf{1}_n) = 0$. Then, there exists $p > 0$ such that $\sum_{i=1}^{n} D_i(p\mathbf{1}_n) < \underline{q}$. Since $D_i \geq 0$ for every i, it follows that $D_i(p\mathbf{1}_n) \leq \underline{q} \leq q_i$ for every i. Therefore, $D(p\mathbf{1}_n) \leq q$ and, by Lemma 3-(c), $q \in X$. □

According to Theorem 1, $CCH(\mathcal{R}(D)) \subseteq X \subseteq CCCH(\mathcal{R}(D))$. This characterizes X up to a set of Lebesgue measure zero. In general, it is not possible to obtain a tighter characterization of X. Here are few examples and counterexamples:

- $n = 1$ and $D(p) = \frac{1}{p}$.
  Since D is not integrable on $[1, \infty)$, by Proposition 1, $X \neq \mathbb{R}_+ = CCCH(\mathcal{R}(D))$. Since $\lim_\infty D = 0$, by Proposition 2, $X = \mathbb{R}_{++} = CCH(\mathcal{R}(D))$.

- $n = 1$ and $D(p) = \frac{1}{p^2}$.
  Since D is integrable on $[1, \infty)$, by Proposition 1, $X = \mathbb{R}_+ = CCCH(\mathcal{R}(D))$, and $X \neq \mathbb{R}_{++} = CCH(\mathcal{R}(D))$.

- $n = 2$ and $D(p) = \left(\frac{1}{p_1}, \frac{1}{p_2^2}\right)$.
  It is easy to see that $X = \mathbb{R}_{++} \times \mathbb{R}_+$, which is neither $CCH(\mathcal{R}(D))$ $\left(= \mathbb{R}_{++}^2\right)$ nor $CCCH(\mathcal{R}(D))$ $\left(= \mathbb{R}_+^2\right)$.

4.2 Continuity and differentiability of u

Continuity. Lemma 10 can be strengthened, provided that consumer surplus is well defined:

Proposition 3. *If $p \in \mathbb{R}_{++} \mapsto \sum_{i=1}^{n} D_i(p\mathbf{1}_n)$ is integrable on $[1, \infty)$, then u is continuous on $X = \mathbb{R}^n_+$.*

Proof. We already know from Lemma 9 and Proposition 1 that u is upper semi-continuous on $\mathbb{R}^n_+$. Assume for a contradiction that u is not lower semi-continuous at point $x \in \mathbb{R}^n_+$. We proceed as in the proof of Lemma 10, but we now have to take care of the fact that some components of x may be equal to zero. There exists $\varepsilon_0 > 0$ such that for every $k \geq 1$, there exists $\tilde{x}^k \in \mathbb{R}^n_+$ such that $\|\tilde{x}^k - x\| < \frac{1}{k}$ and $u(\tilde{x}^k) < u(x) - \varepsilon_0$. By definition of u, for every $k \geq 1$, there exists $p^k \in \mathbb{R}^n_{++}$ such that

$$p^k \cdot \tilde{x}^k + v(p^k) < u(x) - \varepsilon_0.$$

Let $v^k = v(p^k)$ for every $k \geq 1$, and define

$$I = \{i : \ 1 \leq i \leq n \text{ and } x_i \neq 0\}.$$

For every $k \geq 1$, for every $1 \leq i \leq n$, let

$$x_i^k = \begin{cases} \tilde{x}_i^k & \text{if } i \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\|x^k - x\| \leq \|\tilde{x}^k - x\| < \frac{1}{k}$ and

$$\sum_{i \in I} p_i^k x_i^k + v^k = p^k \cdot x^k + v^k < u(x) - \varepsilon_0$$

for every $k \geq 1$. By Proposition 1, $u(0) = \inf_{p>>0} v(p) > -\infty$. Therefore, for every k,

$$\sum_{i \in I} p_i^k x_i^k < u(x) - u(0) - \varepsilon_0.$$

If $x = 0$, then the above inequality yields an immediate contradiction, so suppose $x \neq 0$, and choose $0 < \eta < \min_{i \in I} x_i$. Since $x_i^k \underset{k \to \infty}{\longrightarrow} x_i > 0$ for every $i \in I$, $x_i^k > \eta$ for k high enough. Therefore, for high enough k, $0 < \eta \sum_{i \in I} p_i^k < u(x) - u(0) - \varepsilon_0$. It follows that $\left(\left(p_i^k\right)_{i \in I}\right)_{k \geq 1}$ is bounded. Therefore, there exists $\psi_1 : \mathbb{N}^* \to \mathbb{N}^*$ strictly increasing and $(p_i)_{i \in I} \in \mathbb{R}^I_+$ such that $\left(p_i^{\psi_1(k)}\right)_{i \in I} \underset{k \to \infty}{\longrightarrow} (p_i)_{i \in I}$.

Next, notice that $u(0) < v^{\psi_1(k)} < u(x) - \varepsilon_0$ for every k. Therefore, $\left(v^{\psi_1(k)}\right)_{k \geq 1}$ is bounded and there exists $\psi_2 : \mathbb{N}^* \to \mathbb{N}^*$ strictly increasing and $v \in \mathbb{R}$ such that $v^{\psi_1 \circ \psi_2(k)} \underset{k \to \infty}{\longrightarrow} v$. Let

$\psi = \psi_1 \circ \psi_2$. We know that for every $k \geq 1$,

$$\sum_{i \in I} p_i^{\psi(k)} x_i^{\psi(k)} + v^{\psi(k)} < u(x) - \varepsilon_0.$$

Taking limits, we get:

$$\sum_{i \in I} p_i x_i + v \leq u(x) - \varepsilon_0.$$

Yet, by definition of u, for every $k \geq 1$

$$u(x) \leq p^{\psi(k)} \cdot x + v\left(p^{\psi(k)}\right) = \sum_{i \in I} p_i^{\psi(k)} x_i + v^{\psi(k)}.$$

Taking limits, we obtain that $u(x) \leq \sum_{i \in I} p_i x_i + v$, which is a contradiction. $\square$

Differentiability. We also prove some results on the differentiability of u. Lemma 12 is not, strictly speaking, essential to prove differentiability of u but is of independent interest.

Lemma 12. *Suppose that D is $\mathcal{C}^1$. Assume $\det\left(J(p^0)\right) \neq 0$ at some price vector $p^0 >> 0$, and let $q^0 = D(p^0)$. There exists an open neighborhood of q^0, denoted $\mathcal{N}$, and a $\mathcal{C}^1$ function $P : \mathcal{N} \to \mathbb{R}^n_{++}$ such that for every $p >> 0$ and $q \in \mathcal{N}$, $q = D(p)$ if and only if $p = P(q)$.*

Proof. By the inverse function theorem, there exist open sets $\mathcal{M}$ and $\mathcal{N}$ such that $p^0 \in \mathcal{M}$, $q^0 \in \mathcal{N}$, function $\tilde{D} : p \in \mathcal{M} \mapsto D(p) \in V$ is bijective, and $P \equiv \tilde{D}^{-1}$ is $\mathcal{C}^1$. Let $q \in \mathcal{N}$ and $p \equiv P(q)$. By definition of P, $q = D(p)$. Next, let $\hat{p} \neq p$ in $\mathbb{R}^n_{++}$. We want to show that $D(\hat{p}) \neq q = D(p)$. A sufficient condition for this is that $(p - \hat{p}) \cdot (D(p) - D(\hat{p})) \neq 0$. Notice that, for every $1 \leq i \leq n$,

$$D_i(p) - D_i(\hat{p}) = \int_0^1 \sum_{j=1}^n (p_j - \hat{p}_j) \frac{\partial D_i}{\partial p_j}\left((1-t)\hat{p} + tp\right) dt.$$

Therefore,

$$\begin{aligned}(p - \hat{p}) \cdot (D(p) - D(\hat{p})) &= \int_0^1 \sum_{1 \leq i,j \leq n} \frac{\partial D_i}{\partial p_j}\left((1-t)\hat{p} + tp\right)(p_j - \hat{p}_j)(p_i - \hat{p}_i) dt, \\ &= \int_0^1 (p - \hat{p})\, J\left((1-t)\hat{p} + tp\right)(p - \hat{p})^T\, dt.\end{aligned}$$

Since J and the determinant function are continuous, there exists $0 < \varepsilon < 1$ such that $\det\left(J\left((1-t)\hat{p} + tp\right)\right) \neq 0$ for every $t \in [0, \varepsilon]$. It follows that $J\left((1-t)\hat{p} + tp\right)$ is negative definite for every $t \in [0, \varepsilon]$, and negative semi-definite for every $t \in [\varepsilon, 1]$. Therefore, for every $t \in [0, 1]$, $(p - \hat{p})\, J\left((1-t)\hat{p} + tp\right)(p - \hat{p})^T \leq 0$, and the inequality is strict for every $t \in [0, \varepsilon]$. It follows that $(p - \hat{p}) \cdot (D(p) - D(\hat{p})) < 0$. $\square$

Proposition 4. *Suppose that D is $\mathcal{C}^1$. Assume* $\det(J(p^0)) \neq 0$ *at some price vector* $p^0 >> 0$*, and let $q^0 = D(p^0)$. There exists an open neighborhood of q^0, denoted $\mathcal{N}$, such that u is $\mathcal{C}^1$ on $\mathcal{N}$. Moreover,*

$$\nabla u(q^0) = p^0.$$

Proof. By Lemma 12, there exists an open neighborhood of q^0, denoted $\mathcal{N}$, and a $\mathcal{C}^1$ function $P : \mathcal{N} \to \mathbb{R}^n_{++}$ such that for every $q \in \mathcal{N}$, $D(P(q)) = q$. Let $q \in \mathcal{N}$. Then,

$$\nabla \left. \phi_q(p) \right|_{p=P(q)} = q - D(P(q)) = 0.$$

Therefore, $P(q)$ is a global minimizer of $\phi_q(.)$, and

$$u(q) = \phi_q(P(q)) = P(q) \cdot q + v(P(q)).$$

It follows that u is $\mathcal{C}^1$ on $\mathcal{N}$. In addition, using the envelope theorem,

$$\nabla u(q^0) = \nabla_q \left. \phi_{q^0}(p) \right|_{p=P(q^0)} = P(q^0) = p^0. \qquad \square$$

A special case, often encountered in the industrial organization literature, arises when products are substitutes and total demand ($\sum_{i=1}^n D_i$) is strictly decreasing in every price:

Corollary 2. *Assume that D is $\mathcal{C}^1$, and that, for every $p >> 0$, for every $1 \leq i, j \leq n$ such that $i \neq j$, $\frac{\partial D_i}{\partial p_j}(p) \geq 0$. Assume, in addition, that for every $p >> 0$, for every $1 \leq i \leq n$,*

$$\sum_{j=1}^{n} \frac{\partial D_j}{\partial p_i}(p) < 0.$$

Then, u is $\mathcal{C}^1$ at every point of $\mathcal{R}(D)$.

Proof. Let $p >> 0$. For every $1 \leq i \leq n$,

$$\left| \frac{\partial D_i}{\partial p_i}(p) \right| = -\frac{\partial D_i}{\partial p_i}(p) > \sum_{j \neq i} \frac{\partial D_j}{\partial p_i}(p) = \sum_{j \neq i} \left| \frac{\partial D_j}{\partial p_i}(p) \right|.$$

Therefore, $J(p)$ is strictly diagonally dominant and $\det(J(p)) \neq 0$. By Proposition 4, u is $\mathcal{C}^1$ in an open neighborhood of $D(p)$. Since this holds for every price vector, it follows that u is $\mathcal{C}^1$ at every point of $\mathcal{R}(D)$. $\square$

4.3 A remark on the uniqueness of utility function u

There is a sense in which the utility function we constructed in Section 3.3 is unique "where it matters." Suppose that D can be derived from $(\tilde{X}, \tilde{u})$ and $(\hat{X}, \hat{u})$, and let $\tilde{v}$ and $\hat{v}$ be the corresponding indirect subutility functions. Then, by Lemma 1, there exists a scalar α such that $\tilde{v} = \alpha + \hat{v}$. Moreover, by definition of the indirect subutility functions, for every $p >> 0$,

$$\tilde{u}\left(D(p)\right) = \tilde{v}(p) + p \cdot D(p),$$
$$\hat{u}\left(D(p)\right) = \hat{v}(p) + p \cdot D(p).$$

Therefore, for every $q \in \mathcal{R}(D)$, $\tilde{u}(q) = \alpha + \hat{u}(q)$. This means that, on the range of D, utility functions $\tilde{u}$ and $\hat{u}$ differ by a constant.

This remark implies that, provided that the range of D is nicely shaped (for instance, if it is open and convex), if D can be derived from $(\tilde{X}, \tilde{u})$, then $\tilde{u}$ inherits all the regularity, monotonicity and convexity properties of u on $\mathcal{R}(D)$. For instance, if $\mathcal{R}(D) = \mathbb{R}^n_{++}$ and D can be derived from $(\tilde{X}, \tilde{u})$, then $\tilde{u}$ is increasing, convex and continuous on $\mathbb{R}^n_{++}$. If, in addition, products are substitutes and total demand is strictly decreasing in every price, then $\tilde{u}$ is $\mathcal{C}^1$ on $\mathbb{R}^n_{++}$.

4.4 Measurement errors

In this section, we study quasi-linear integrability in the presence of measurement errors. In doing so, we follow closely the approach proposed by Lewbel (2001) for the case without an outside good. The demand function of a consumer with observable attributes a and unobservable attributes φ is $p \to d(p, a, \varphi)$. The conditional probability distribution of the unobservable attributes φ given (p, a) is assumed to have a compact support and a continuous density. We also assume that the conditional probability distribution is independent of p. Intuitively, we view the unobservable consumer attributes φ as preference parameters, which, in line with classical demand theory, are not affected by prices. Finally, we assume that d is continuous in (p, φ) for every a.

Next, we build a statistical model of consumer demand. For every (p, a), define

$$D(p, a) = E\left(d(p, a, \varphi)|(p, a)\right) = \int d(p, a, \varphi) f(\varphi|a) d\varphi,$$

where $f(\cdot|a)$ is the conditional density of φ given a. Then,

$$d(p, a, \varphi) = D(p, a) + \left(d(p, a, \varphi) - D(p, a)\right) \equiv D(p, a) + \varepsilon(p, a, \varphi),$$

and, by definition, $E(\varepsilon(p, a, \varphi)|(p, a)) = 0$. This gives rise to the following econometric model:

$$q^i = D(p^i, a^i) + \varepsilon^i, \quad E(\varepsilon^i|p^i, a^i) = 0,$$

where the vector q^i is consumer i's demand for each product, p^i is the vector of prices consumer i has access to, and a^i is consumer i's observable attributes. Using Lewbel (2001)'s terminology, D is the statistical, or econometric, demand function, whereas d is the economic demand function.

Given the assumptions made above, it is easily seen that, if the economic demand function

$d(\cdot, a, \varphi)$ is quasi-linearly integrable for every φ, then the statistical demand function $D(\cdot, a)$ is quasi-linearly integrable. This follows because $D(\cdot, a)$ is continuous and, if, e.g., condition (ii) in Theorem 1 holds for $d(\cdot, a, \varphi)$ for every φ, then it also holds for $D(\cdot, a)$.[8]

Suppose that an econometrician is interested in finding out whether the behavior of consumers with observable attributes a can be rationalized by a quasi-linear utility function. The econometrician would first obtain an estimate of the statistical demand function $\hat{D}$. He would then use condition (ii), (iii) or (iv) in Theorem 1 to test the null hypothesis that $D(\cdot, a)$ is quasi-linearly integrable against the alternative hypothesis that it is not. By rejecting the null hypothesis, the econometrician would provide statistical evidence that a positive mass of consumers with observable attributes a behave in a way which is inconsistent with the maximization of a quasi-linear utility function.

The statistical demand function can also be used to evaluate the average consumer surplus effects of price changes. Suppose that $d(\cdot, a, \varphi)$ is quasi-linearly integrable for every φ, and let C be a continuously differentiable path going from the original price vector to the new price vector. Then, the average variation in money-metric consumer welfare for consumers with attribute a is given by

$$\int \left(\int_C d(p, a, \varphi) \cdot dp \right) f(\varphi|a) d\varphi = \int_C \left(\int d(p, a, \varphi) f(\varphi|a) d\varphi \right) \cdot dp = \int_C D(p, a) \cdot dp.$$

Therefore, the average variation in consumer surplus for consumers with observable attributes a is equal to the variation in consumer surplus associated with the statistical demand function $D(\cdot, a)$.

5 Relationship to the Rationalizability Literature

Complementary to the integrability approach, there is also a literature on rationalizability, pioneered by Afriat (1967). That literature is concerned with the relationship between preference maximization and various axioms of choice.

Brown and Calsamiglia (2007) derive necessary and sufficient conditions for a finite data set $(p^i, d^i)_{i \in \mathcal{I}} \in (R^n_{++} \times \mathbb{R}^n_+)^{\mathcal{I}}$ to be derivable from the maximization of a quasi-linear utility function. They find that $(p^i, d^i)_{i \in \mathcal{I}}$ is quasi-linearly rationalizable if and only if for every $\{r_1, r_2, \ldots, r_m\} \subseteq \mathcal{I}$,

$$p^{r_1} \cdot (d^{r_2} - d^{r_1}) + p^{r_2} \cdot (d^{r_3} - d^{r_2}) + \ldots + p^{r_m} \cdot (d^{r_1} - d^{r_m}) \geq 0.$$

This is equivalent to

$$(p^{r_2} - p^{r_1}) \cdot d^{r_1} + (p^{r_3} - p^{r_2}) \cdot d^{r_2} + \ldots + (p^{r_1} - p^{r_m}) \cdot d^{r_m} \geq 0$$

[8] Of course, the converse is not true in general, i.e., $D(\cdot, a)$ could be quasi-linearly integrable even if $d(\cdot, \varphi, a)$ is not quasi-linearly integrable for some φ's.

for every $\{r_1, r_2, \ldots, r_m\} \subseteq \mathcal{I}$, which is the finite-data counterpart of our condition (1).

It is sometimes argued that rationalizability results with finite data are more useful than those with infinite data, because, in practice, applied researchers never have access to infinite data (see, e.g., Varian, 1982, 1983). An econometrician who has access to data on the same consumer or household making choices from multiple budget sets could indeed use Brown and Calsamiglia (2007)'s condition to test whether that consumer's choices can be rationalized by a quasi-linear utility function.[9] However, applied researchers often do not have access to such longitudinal data, and therefore need to rely on data for a cross-section of households, with each household making a choice from a single budget set. While it would in principle be possible to test whether these cross-sectional data satisfy a certain axiom of rationality, such a test seems implausibly restrictive, since consumers are likely to have heterogeneous preferences, even after controlling for observable consumer attributes.[10] For this reason, when longitudinal data are not available, it seems more reasonable to build a statistical model with individual-specific, additive error terms, like the one we described in Section 4.4, and test whether the statistical demand function is integrable by exploring the properties of its Slutsky matrix (see Lewbel, 2001, for references).

In a recent paper, Nishimura, Ok, and Quah (2016) develop a general approach to revealed preference theory in environments that go beyond the classical setting of consumer theory. The starting point is a choice environment $((X, \trianglerighteq), \mathcal{A})$, where $(X, \trianglerighteq)$ is a preordered set and $\mathcal{A} \subseteq 2^X \setminus \{\emptyset\}$. Nishimura, Ok, and Quah (2016) study both the finite-data case ($|\mathcal{A}| < \infty$) and the infinite-data case ($|\mathcal{A}| = \infty$). The idea behind the preorder $\trianglerighteq$ is that, if $x \trianglerighteq y$, then any individual in the society prefers x to y. The authors derive conditions under which a choice correspondence $c : \mathcal{A} \rightrightarrows X$ (such that $c(A) \subseteq A$ for every A) can be rationalized (resp. weakly rationalized) by a preference relation $\succsim$ that extends $\trianglerighteq$.[11] They find that $\trianglerighteq$-rationalizability is equivalent to a generalized version of Afriat (1967)'s cyclical consistency axiom, which they call the $\trianglerighteq$-cyclical consistency axiom, and that strict $\trianglerighteq$-rationalizability is equivalent to a generalized version of Richter (1966)'s congruence axiom, called the $\trianglerighteq$-congruence axiom.

We can apply Nishimura, Ok, and Quah (2016)'s results to our setting as follows. The set of objects of choice is $X = \mathbb{R}^{n+1}_+$. The exogenously given dominance relation $\trianglerighteq$ is defined as follows: $(q_0, q) \trianglerighteq (q_0', q)$ if and only if $q_0 \geq q_0'$, for every $q_0, q_0' \geq 0$ and $q \in \mathbb{R}^n_+$. The set $\mathcal{A}$ is defined as $\mathcal{A} = \{B(p, y)\}_{(p,y) \in \mathcal{K}}$, where

$$\mathcal{K} = \left\{(p, y) \in \mathbb{R}^n_{++} \times \mathbb{R}_+ : \ p \cdot D(p) \leq y \right\},$$

[9] In the presence of multiple households, the econometrician can simply check that condition separately for each household, thereby allowing for arbitrary heterogeneity.

[10] A similar problem arises with longitudinal data when there are measurement errors.

[11] $\succsim$ weakly (resp. strictly) rationalizes c if for every A, $c(A)$ is contained in (resp. equal to) the set of x's that maximize $\succsim$ in A.

and, for every $(p, y) \in \mathcal{K}$,

$$B(p,y) = \{(q_0, q) \in X : \ q_0 + p \cdot q \leq y\}.$$

The choice correspondence c is: For every $B(p, y) \in \mathcal{A}$,

$$c\,(B(p,y)) = \{(y - p \cdot D(p), D(p))\}.$$

In the following, we say that D satisfies the $\trianglerighteq$-cyclical consistency (resp. $\trianglerighteq$-congruence) axiom if c satisfies the $\trianglerighteq$-cyclical consistency (resp. $\trianglerighteq$-congruence) axiom. Similarly, we say that D is weakly (resp. strictly) $\trianglerighteq$-rationalizable if c is weakly (resp. strictly) $\trianglerighteq$-rationalizable. We prove the following proposition:

Proposition 5. *Let $D : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}^n_+$ be a demand system:*

(a) D satisfies the $\trianglerighteq$-cyclical consistency axiom if and only if condition (1) *holds for every finite collection of prices. Therefore, D can be weakly rationalized by a preference relation $\succsim$ that is strictly monotone in the consumption of the outside good if and only if condition* (1) *holds for every finite collection of price vectors.*

(b) D satisfies the $\trianglerighteq$-congruence axiom if and only if, for every $k \geq 2$ and $(p^i)_{1 \leq i \leq k} \in (\mathbb{R}^n_{++})^k$, condition (1) *holds, and*

$$\begin{array}{c} (p^2 - p^1) \cdot D(p^1) + (p^3 - p^2) \cdot D(p^2) + \ldots + (p^1 - p^k) \cdot D(p^k) = 0, \\ \Updownarrow \\ D(p^1) = D(p^2) = \ldots = D(p^k) \end{array} \tag{3}$$

Therefore, D can be strictly rationalized by a preference relation $\succsim$ that is strictly monotone in the consumption of the outside good if and only if conditions (1) *and* (3) *hold for every finite collection of prices.*

(c) If D is continuous, then the $\trianglerighteq$-cyclical consistency axiom and the $\trianglerighteq$-congruence axiom are equivalent.

Proof.

(a). In our setting, the $\trianglerighteq$-cyclical consistency axiom can be rewritten as follows: For every $k \geq 2$, $(p^i)_{1 \leq i \leq k}$, and $(y^i)_{1 \leq i \leq k}$, if $(p^2 - p^1) \cdot D(p^1) \leq y^2 - y^1$, $(p^3 - p^2) \cdot D(p^2) \leq y^3 - y^2$, $\ldots$, and $(p^1 - p^k) \cdot D(p^k) \leq y^1 - y^k$, then, $(p^2 - p^1) \cdot D(p^1) = y^2 - y^1$, $(p^3 - p^2) \cdot D(p^2) = y^3 - y^2$, $\ldots$, and $(p^1 - p^k) \cdot D(p^k) = y^1 - y^k$.

Suppose that the $\trianglerighteq$-congruence axiom does not hold. Then, there exist $k \geq 2$, $(p^i)_{1 \leq i \leq k}$, and $(y^i)_{1 \leq i \leq k}$ such that $(p^2 - p^1) \cdot D(p^1) \leq y^2 - y^1$, $\ldots$, $(p^1 - p^k) \cdot D(p^k) \leq y^1 - y^k$ with at least one strict inequality. It follows that

$$(p^2 - p^1) \cdot D(p^1) + \ldots + (p^1 - p^k) \cdot D(p^k) < y^2 - y^1 + \ldots + y^1 - y^k = 0,$$

i.e., condition (1) does not hold for $(p^i)_{1 \le i \le k}$.

Conversely, suppose that condition (1) does not hold for some $(p^i)_{1 \le i \le k}$. As in the proof of Lemma 11, choose a high enough x, and define $y^i = x + \sum_{j=1}^{i-1} (p^{j+1} - p^j) \cdot D(p^j)$ for every i. Then, for every $1 \le i \le k-1$, $(p^{i+1} - p^i) \cdot D(p^i) = y^{i+1} - y^i$, and $(p^1 - p^k) \cdot D(p^k) < y^1 - y^k$. Therefore, the $\trianglerighteq$-cyclical consistency axiom does not hold.

(b). By Proposition 6 in Nishimura, Ok, and Quah (2016), D satisfies the $\trianglerighteq$-congruence axiom if and only if it satisfies the $\trianglerighteq$-cyclical consistency axiom, and, for every every $k \ge 2$, $(p^i)_{1 \le i \le k}$, and $(y^i)_{1 \le i \le k}$, $(p^2 - p^1) \cdot D(p^1) \le y^2 - y^1$, $(p^3 - p^2) \cdot D(p^2) \le y^3 - y^2$, $\ldots$, and $(p^1 - p^k) \cdot D(p^k) \le y^1 - y^k$ implies that

$$\left(y^1 - p^1 \cdot D(p^1), D(p^1)\right) = \left(y^2 - p^2 \cdot D(p^2), D(p^2)\right) = \ldots = \left(y^k - p^k \cdot D(p^k), D(p^k)\right).$$

It is straightforward to adapt the argument used in part (a) to show that this is equivalent to conditions (1) and (3) being satisfied for every finite collection of prices.

(c). Assume that D is continuous and satisfies $\trianglerighteq$-cyclical consistency. Then, by part (a) and Theorem 1, D is quasi-linearly integrable. Let (X, u) such that D can be derived from (X, u) (and recall from Definition 3 that (X, u) provides a strict rationalization). Let $\succsim$ be the preference relation produced by (X, u). Let $k \ge 2$, $(p^i)_{1 \le i \le k}$, and $(y^i)_{1 \le i \le k}$, such that $(p^2 - p^1) \cdot D(p^1) \le y^2 - y^1$, $\ldots$, $(p^1 - p^k) \cdot D(p^k) \le y^1 - y^k$. Then, for every $1 \le i \le k$,[12]

$$\left(y^{i+1} - p^{i+1} \cdot D(p^{i+1}), D(p^{i+1})\right) \succsim \left(y^i - p^i \cdot D(p^i), D(p^i)\right).$$

Therefore, by transitivity, for every i,

$$\left(y^{i+1} - p^{i+1} \cdot D(p^{i+1}), D(p^{i+1})\right) \sim \left(y^i - p^i \cdot D(p^i), D(p^i)\right).$$

Suppose that $(y^{i+1} - p^{i+1} \cdot D(p^{i+1}), D(p^{i+1})) \neq (y^i - p^i \cdot D(p^i), D(p^i))$ for some i. Then, since $(y^i - p^i \cdot D(p^i), D(p^i))$ is feasible at price vector p^{i+1} and income level y^{i+1}, and by definition of $\succsim$, it follows that

$$\left(y^{i+1} - p^{i+1} \cdot D(p^{i+1}), D(p^{i+1})\right) \succ \left(y^i - p^i \cdot D(p^i), D(p^i)\right),$$

which is a contradiction. Therefore, D satisfies the $\trianglerighteq$-congruence axiom. □

Parts (a) and (b) allow us to obtain a rationalizing preference relation $\succsim$ when D is not continuous, a case Theorem 1 cannot handle. If D is continuous, then Proposition 5

[12] The index i is taken modulo k, so that $k + 1 = 1$.

is significantly weaker than Theorem 1 in the following sense: The preference relation $\succsim$ delivered by Proposition 5 (1) may or may not be quasi-linear, (2) may or may not have a utility representation, (3) may or may not be convex, (4) may or may not be monotone, (5) may or may not be continuous.[13] Moreover, (6) Nishimura, Ok, and Quah (2016)'s approach is based on the axiom of choice, and, hence, non-constructive. Our Theorem 1 provides a strict improvement along all those dimensions.

Finally, by part (c), if demand is continuous, then the $\trianglerighteq$-cyclical consistency axiom and the $\trianglerighteq$-congruence axiom are equivalent. This is not necessarily true when demand is not continuous. To see this, consider the following counterexample with $n = 2$:

$$D(p_1, p_2) = \begin{cases} (1, 0) & \text{if } p_1 < p_2, \\ (0, 1) & \text{if } p_1 > p_2, \\ (\alpha(p_1), 1 - \alpha(p_1)) & \text{if } p_1 = p_2, \end{cases}$$

where $\alpha(x) \equiv x/(1 + x)$ for all $x > 0$. Note that D can be weakly $\trianglerighteq$-rationalized by the following utility function:

$$\forall (q_0, q_1, q_2) \in \mathbb{R}^3_+, \quad U(q_0, q_1, q_2) = \begin{cases} \mathrm{e}^{q_0} & \text{if } q_1 + q_2 \geq 1, \\ -\mathrm{e}^{-q_0} & \text{if } q_1 + q_2 < 1. \end{cases}$$

Therefore, D satisfies the $\trianglerighteq$-cyclical consistency axiom. Note however that, for every $x, x' > 0$ such that $x \neq x'$, we have that $D(x', x') \neq D(x, x)$, but

$$((x', x') - (x, x)) \cdot (D(x', x') - D(x, x)) = (x' - x, x' - x) \cdot (\alpha(x') - \alpha(x), \alpha(x) - \alpha(x')) = 0.$$

Therefore, D does not satisfy the $\trianglerighteq$-congruence axiom.

A Appendix: Technical Lemmas

The following lemmas are well known, but, since they play an important role in our analysis, we provide short proofs of them.

Lemma 13. *Let $D : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}^n_+$ be a continuous function. Then, D has a potential (i.e., there exists a function v such that $\nabla v = -D$) if and only if D is conservative (i.e., for every closed and piecewise-$\mathcal{C}^1$ path C, $\int_C D(p) \cdot dp = 0$). In addition, if D is $\mathcal{C}^1$, then D has a potential if and only if the matrix $J(p) = \left(\frac{\partial D_i}{\partial p_j}(p)\right)_{1 \leq i,j \leq n}$ is symmetric for every $p >> 0$.*

[13] One way of addressing concerns (2) and (5) would be to apply Nishimura, Ok, and Quah (2016)'s Theorem 4, which delivers a continuous preference relation, and hence, a continuous utility representation. In order to apply that theorem, one would first need to prove that the transitive closure of the union of the direct revealed preference relation and the preorder $\trianglerighteq$ is a continuous preorder, which seems impractical in our framework.

Moreover, v is a potential for D if and only if there exists $\alpha \in \mathbb{R}$ such that

$$v(p) = \alpha - \int_0^1 (p - \mathbf{1}_n) \cdot D\left(tp + (1-t)\mathbf{1}_n\right) dt, \ \forall p >> 0. \tag{4}$$

Proof. Suppose that D has a potential, i.e., there exists $v : \mathbb{R}_{++} \longrightarrow \mathbb{R}$ such that $\nabla v = -D$. Then, for every closed and piecewise-$\mathcal{C}^1$ path C starting at point $p^0 >> 0$,

$$\int_C D(p) \cdot dp = v(p^0) - v(p^0) = 0.$$

Moreover, if D is $\mathcal{C}^1$, then v has continuously differentiable partial derivatives. Therefore, v is $\mathcal{C}^2$, and, by Schwarz's theorem, $J(p)$ is symmetric for every p.

Conversely, suppose that D is conservative. Define the function v as in equation (4) with $\alpha = 0$. Since D is conservative, we have that, for every $p, p' >> 0$,

$$\int_0^1 (p - \mathbf{1}_n) \cdot D\left(tp + (1-t)\mathbf{1}_n\right) dt + \int_0^1 (p' - p) \cdot D\left(tp' + (1-t)p\right) dt \\ + \int_0^1 (\mathbf{1}_n - p') \cdot D\left(t\mathbf{1}_n + (1-t)p'\right) dt = 0,$$

i.e.,

$$v(p') - v(p) = -(p' - p) \cdot \int_0^1 D\left(tp' + (1-t)p\right) dt.$$

It follows that

$$\frac{v(p') - v(p) + (p' - p) \cdot D(p)}{\|p' - p\|} = \underbrace{\frac{p' - p}{\|p' - p\|}}_{\text{bounded}} \cdot \underbrace{\int_0^1 \left(D(p) - D\left(tp' + (1-t)p\right)\right) dt}_{\underset{p' \to p}{\longrightarrow} 0, \text{ by continuity of } D}.$$

Therefore, $\nabla v = -D$, and D has a potential.

Next, suppose that D is $\mathcal{C}^1$ and $J(p)$ is symmetric for every p. To ease notation, let $r(p,t) = \mathbf{1}_n + t(p - \mathbf{1}_n)$, and define the function v as in equation (4) with $\alpha = 0$. For every j in $\{1, \ldots, n\}$,

$$\begin{aligned} \frac{\partial v}{\partial p_j} &= -\int_0^1 D_j\left(r(p,t)\right) dt - \int_0^1 t \sum_{i=1}^n (p_i - 1) \frac{\partial D_i}{\partial p_j}\left(r(p,t)\right) dt, \\ &= -\int_0^1 D_j\left(r(p,t)\right) dt - \int_0^1 t \sum_{i=1}^n (p_i - 1) \frac{\partial D_j}{\partial p_i}\left(r(p,t)\right) dt, \\ &= -\int_0^1 D_j\left(r(p,t)\right) dt - \left(\Big[tD_j\left(r(p,t)\right) \Big]_0^1 - \int_0^1 D_j\left(r(p,t)\right) dt \right), \\ &= -D_j(p), \end{aligned}$$

where the second equality follows from the symmetry of J and the third equality follows by integration by parts. Therefore, $\nabla v = -D$, and D has a potential.

Finally, suppose that v and w are such that $\nabla v = \nabla w = -D$. Let $f = v - w$. Then, $\nabla f = 0$. Since $\mathbb{R}^n_{++}$ is convex, we can apply Theorem 9.19 in Rudin (1976) to conclude that f is constant. This shows that v is a potential for D if and only if v can be written as in equation (4). □

Lemma 14. *Let $v : \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$ function. Define $D = -\nabla v$. Then, v is convex if and only if D satisfies the law of demand.*

Proof. Let $p, p' >> 0$. Define the function $f(t) = v(p + t(p' - p))$ on interval $[0, 1]$, and note that

$$f'(t) = -(p' - p) \cdot D(p + t(p' - p)).$$

Assume that v is convex. Then, for every $t, t', \lambda \in [0, 1]$,

$$\begin{aligned} f(\lambda t + (1 - \lambda)t') &= v\left(\lambda(p + t(p' - p)) + (1 - \lambda)(p + t'(p' - p))\right), \\ &\leq \lambda f(t) + (1 - \lambda) f(t'), \end{aligned}$$

so f is convex as well. It follows that f' is non-decreasing. In particular,

$$0 \leq f'(1) - f'(0) = -(p' - p) \cdot (D(p') - D(p)).$$

Therefore, D satisfies the law of demand.

Conversely, suppose that D satisfies the law of demand. Then, for every $t, t' \in [0, 1]$ such that $t' > t$,

$$f(t') - f(t) = -(p' - p) \cdot (D(p + t'(p' - p)) - D(p + t(p' - p))) \geq 0.$$

Therefore, f is convex, and for every $t \in [0, 1]$,

$$v((1 - t)p + tp') = f(t) \leq (1 - t)f(0) + tf(1) = (1 - t)v(p) + tv(p').$$

Therefore, v is convex. □

References

Afriat, S. N. (1967): "The Construction of Utility Functions from Expenditure Data," *International Economic Review*, 8(1), 67–77.

Amir, R., P. Erickson, and J. Jin (2017): "On the Microeconomic Foundations of Linear Demand for Differentiated Products," *Journal of Economic Theory*, forthcoming.

Brown, D. J., and C. Calsamiglia (2007): "The Nonparametric Approach to Applied Welfare Analysis," *Economic Theory*, 31(1), 183–188.

Dudley, R. M. (1999): *Uniform Central Limit Theorems.* Cambridge University Press, Cambridge.

Dziewulski, P., and J. Quah (2014): "Testing for Production with Complementarities," Economics Series Working Papers 722, University of Oxford, Department of Economics.

Hosoya, Y. (2016): "Revealed Preference Theory and the Slutsky Matrix," *Mita Journal of Economics*, 108(3), 45–60.

Hurwicz, L., and H. Uzawa (1971): "On the Integrability of Demand Functions," in *Preferences, Utility, and Demand*, ed. by J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein. Harcourt Brace Jovanovich, New York.

Jackson, M. O. (1986): "Integration of Demand and Continuous Utility Functions," *Journal of Economic Theory*, 38(2), 298–312.

Krishna, V., and H. F. Sonnenschein (1990): "Duality in Consumer Theory," in *Preferences, Uncertainty and Optimality*, ed. by J. S. Chipman, D. McFadden, and M. K. Richter. Westview Press, Boulder.

Lewbel, A. (2001): "Demand Systems with and without Errors," *The American Economic Review*, 91(3), 611–618.

Nishimura, H., E. A. Ok, and J. Quah (2016): "A Comprehensive Approach to Revealed Preference Theory," .

Richter, M. K. (1966): "Revealed Preference Theory," *Econometrica*, 34(3), 635–645.

Sákovics, J. (2013): "Revealed Cardinal Preference," *Economic Theory Bulletin*, 1(1), 39–45.

Varian, H. R. (1982): "The Nonparametric Approach to Demand Analysis," *Econometrica*, 50(4), 945–973.

——— (1983): "Non-Parametric Tests of Consumer Behaviour," *The Review of Economic Studies*, 50(1), 99–110.

Vives, X. (2000): *Oligopoly Pricing: Old Ideas and New Tools.* Cambridge, MA: MIT Press.