Appendix to
An Aggregative Games Approach to Merger Analysis in Multiproduct-Firm Oligopoly

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I Proof of Proposition 1: Necessity and Sufficiency of First-Order Conditions

Proof. Fix a profile of prices p^{-f} for firm f's rivals, and let $\mathcal{N}^f = \bigcup_{l \in f} l$. Define

$$H^{0\prime} = H^0 + \sum_{g \in \mathcal{F} \setminus \{f\}} \sum_{l \in g} \left(\sum_{j \in l} h_j(p_j^{-f}) \right)^{\beta} > 0,$$

and

$$G(p) = \beta \frac{\sum_{l \in f} \left(\sum_{i \in l} h_i(p_i) \right)^{\beta - 1} \sum_{j \in l} (p_j - c_j)(-h_j'(p_j))}{H^{0\prime} + \sum_{l \in f} \left(\sum_{i \in l} h_i(p_i) \right)^{\beta}},$$

for every profile of prices $p = (p_j)_{j \in \mathcal{N}^f}$. Note that $G(p)$ is the profit firm f receives when it sets the price vector p and its rivals rivals set the price vector p^{-f}. Our goal is to show that the maximization problem

$$\max_{p \in \mathbb{R}_{++}^{\mathcal{N}^f}} G(p)$$

has a unique solution, and that the price vector p solves that maximization problem if and only if it satisfies the first-order conditions.

The proof follows a similar development as the proof of Lemmas B–H in the Appendix of Nocke and Schutz (2018). It proceeds as follows. We first show that pricing some (or all) of the products below cost is strictly suboptimal (Step 1). We then extend the domain of G to price vectors that have infinite components (Step 2). Combining Steps 1 and 2 allows us to show that the profit maximization problem has a solution (Step 3). We then show that

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there exists a unique price vector satisfying the first-order conditions of profit maximization (Step 4). Combining Steps 1–4, we can conclude that the profit maximization problem has a unique solution, and that first-order conditions are necessary and sufficient for optimality.

Step 1: No product is priced below cost. We first argue that firm f's products are substitutes. Let $n, n' \in f$ and $(i, i') \in n \times n'$ such that $i \neq i'$. If $n \neq n'$, then

$$\frac{\partial D_i}{\partial p_{i'}} = \beta^2 \frac{h'_i H_n^{\beta-1} h'_{i'} H_{n'}^{\beta-1}}{H^2} > 0.$$

If instead $n = n'$, then

$$\frac{\partial D_i}{\partial p_{i'}} = \frac{\beta h'_i h'_{i'}}{H} \left((1-\beta) H_n^{\beta-2} + \beta \frac{H_n^{2(\beta-1)}}{H} \right) > 0.$$

Let p be a price vector for firm f such that $p_j < c_j$ for some product $j \in \mathcal{N}^f$. Define a new price vector $\tilde{p}$ for firm f such that for every $i \in \mathcal{N}^f$, $\tilde{p}_i = \max(c_i, p_i)$. When firm f deviates from p to $\tilde{p}$, it stops making losses on those products that were originally priced below cost, and, by substitutability, it makes more profits on those products that were priced above cost. Therefore, price vector p is not optimal for firm f. When looking for a solution to firm f's profit maximization problem, we can therefore confine our attention to price vectors in $\prod_{j \in \mathcal{N}^f} [c_j, \infty)$.

Step 2: Defining G at infinite prices. Let $\hat{p} \in \prod_{j \in \mathcal{N}^f} [c_j, \infty]$. Suppose $\hat{p}$ has at least one infinite component, and let $(p^k)_{k \geq 0}$ be a sequence over $\prod_{j \in \mathcal{N}^f} [c_j, \infty)$ such that $p^k \underset{k \to \infty}{\longrightarrow} \hat{p}$. Let

$$f' = \{l \in f : \exists i \in l \text{ s.t. } \hat{p}_i < \infty\}$$

and

$$\mathcal{N}^{f\prime} = \{j \in \mathcal{N}^f : \hat{p}_j < \infty\}.$$

Clearly, as k tends to infinity, the denominator of $G(p^k)$ tends to[1]

$$H^{0\prime} + \sum_{l \in f'} \left(\sum_{j \in l \cap \mathcal{N}^{f\prime}} h_j(\hat{p}_j) \right)^{\beta}.$$

Next, let $i \in \mathcal{N}^f \setminus \mathcal{N}^{f\prime}$. Let $l \in f$ be the nest that contains product i. Note that, for every $k \geq 0$,

$$(p_i^k - c_i)(-h'_i(p_i^k)) \left(\sum_{j \in l} h_j(p_j^k) \right)^{\beta-1} \leq (p_i^k - c_i)(-h'_i(p_i^k)) \left(h_i(p_i^k) \right)^{\beta-1}.$$

[1]By convention, the sum of an empty collection of reals is zero.

Under NCES demand,

$$(p_i^k - c_i)(-h_i'(p_i^k))\left(h_i(p_i^k)\right)^{\beta-1} \leq (\sigma-1)a_i(p_i^k)^{\beta(1-\sigma)} \underset{k\to\infty}{\longrightarrow} 0.$$

Under NMNL demand,

$$(p_i^k - c_i)(-h_i'(p_i^k))\left(h_i(p_i^k)\right)^{\beta-1} \leq \frac{1}{\lambda}p_i^k \exp\left(\frac{\beta}{\lambda}\left(a_i - p_i^k\right)\right) \underset{k\to\infty}{\longrightarrow} 0.$$

It follows that

$$G(p^k) \underset{k\to\infty}{\longrightarrow} \beta\frac{\sum_{l\in f'}\left(\sum_{i\in l\cap\mathcal{N}^{f'}} h_i(\hat{p}_i)\right)^{\beta-1}\sum_{j\in l\cap\mathcal{N}^{f'}}(\hat{p}_j - c_j)(-h_j'(\hat{p}_j))}{H^{0\prime} + \sum_{l\in f'}\left(\sum_{i\in l\cap\mathcal{N}^{f'}} h_i(\hat{p}_i)\right)^{\beta}} \equiv G(\hat{p}).$$

We have thus extended the domain of G to $\prod_{j\in\mathcal{N}^f}[c_j,\infty]$. Note that, at $\hat{p}$, G has smooth partial derivatives with respect to $(p_i)_{i\in\mathcal{N}^{f'}}$.

Step 3: The profit maximization problem has a solution. By continuity of G (as established in the previous step) and compactness of $\prod_{j\in\mathcal{N}^f}[c_j,\infty]$, the maximization problem

$$\max_{p\in\prod_{j\in\mathcal{N}^f}[c_j,\infty]} G(p)$$

has a solution $\hat{p}$. Clearly, $\hat{p}$ has at least one finite component, for otherwise $G(\hat{p})$ would be equal to zero, as shown above.

Assume for a contradiction that $\hat{p}$ has some infinite components, and define f' and $\mathcal{N}^{f'}$ as in the previous step. Since $\hat{p}$ maximizes G, it must be the case that $\left.\frac{\partial G}{\partial p_i}\right|_{\hat{p}} = 0$ for every $i \in \mathcal{N}^{f'}$. Manipulating the first order conditions as we did in Section 2.3, we obtain the existence of a $\tilde{\mu}^f$ such that, for every $i \in \mathcal{N}^{f'}$,

$$\frac{\hat{p}_i - c_i}{\hat{p}_i}\frac{\hat{p}_i h_i''(\hat{p}_i)}{-h_i'(\hat{p}_i)} = \tilde{\mu}^f.$$

Under NCES, $(\hat{p}_i h_i''(\hat{p}_i))/(-h_i'(\hat{p}_i)) = \sigma$, so that $\tilde{\mu}^f < \sigma$. Moreover, under both NCES and NMNL demand, $\tilde{\mu}^f$ satisfies

$$\tilde{\mu}^f\left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \tilde{\alpha}\beta\tilde{\mu}^f\frac{\sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}{H^{0\prime} + \sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}, \quad \text{(i)}$$

so that $\tilde{\mu}^f > 1$.

Fix a product $i \in \mathcal{N}^f \setminus \mathcal{N}^{f'}$, and let $n \in f$ be the nest that contains product i. For every $x \geq c_i$, let $\tilde{G}(x)$ be the value of G when product i is priced at x and all the other products are priced according to $\hat{p}$. We showed in the previous step that $\tilde{G}(x) \underset{x\to\infty}{\longrightarrow} G(\hat{p})$. Note that, for every $x \in (c_i, \infty)$,

$$\tilde{G}'(x) = D_i \times \Bigg(1 - (x-c_i)\frac{h_i''(x)}{-h_i'(x)} + (1-\beta)\frac{(x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f \sum_{j\in(n\cap\mathcal{N}^{f'})\setminus\{i\}} h_j(\hat{p}_j)}{h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f'})\setminus\{i\}} h_j(\hat{p}_j)}$$
$$+\beta\frac{\left(h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f'})\setminus\{i\}} h_j(\hat{p}_j)\right)^{\beta-1}\left((x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f \sum_{j\in(n\cap\mathcal{N}^{f'})\setminus\{i\}} h_j(\hat{p}_j)\right)}{H^{0\prime} + \left(h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f'})\setminus\{i\}} h_j(\hat{p}_j)\right)^{\beta} + \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}$$
$$+\beta\tilde{\alpha}\tilde{\mu}^f \frac{\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}{H^{0\prime} + \left(h_i(x) + \sum_{j\in(n\cap\mathcal{N}^{f'})\setminus\{i\}} h_j(\hat{p}_j)\right)^{\beta} + \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}\Bigg), \quad \text{(ii)}$$

where we have used the simplification derived in equation (4).

We argue that $\tilde{G}'(x) < 0$ for x sufficiently high. We distinguish two cases. Assume first that $n \notin f'$, i.e., $\hat{p}_j = \infty$ for every $j \in n$. Then, $\tilde{G}'(x)$ simplifies to

$$\tilde{G}'(x) = D_i \Bigg(1 - (x-c_i)\frac{h_i''(x)}{-h_i'(x)} + (1-\beta)(x-c_i)\frac{-h_i'(x)}{h_i(x)}$$
$$+\beta\frac{h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) + \tilde{\alpha}\tilde{\mu}^f \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}{H^{0\prime} + h_i(x)^{\beta} + \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}\Bigg). \quad \text{(iii)}$$

Under NCES demand, $(x-c_i)\frac{h_i''(x)}{-h_i'(x)}$ and $(x-c_i)\frac{-h_i'(x)}{h_i(x)}$ tend to σ and $\sigma - 1$, respectively, as x goes to infinity, whereas

$$h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) = (\sigma-1)a_i x^{\beta(1-\sigma)}\frac{x-c_i}{x}$$

tends to zero. It follows that the term in parenthesis in equation (iii) tends to

$$1 - \sigma + (1-\beta)(\sigma-1) + \beta\tilde{\alpha}\tilde{\mu}^f \frac{\sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}}{H^{0\prime} + \sum_{l\in f'\setminus\{n\}}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^{\beta}},$$

which, using equation (i), simplifies to

$$\begin{aligned} -\beta(\sigma-1) + \tilde{\mu}^f(1 - \tilde{\alpha}(1-\beta)) - 1 &< -\beta(\sigma-1) + \sigma(1-\tilde{\alpha}(1-\beta)) - 1, \\ &= \frac{1}{1-\tilde{\alpha}}\Big(-\beta\tilde{\alpha} + (1-\tilde{\alpha}(1-\beta)) - (1-\tilde{\alpha})\Big), \\ &= 0. \end{aligned}$$

Hence, $\tilde{G}'(x) < 0$ for high enough x.

Under NMNL demand,

$$h_i(x)^{\beta-1}(x-c_i)(-h_i'(x)) = \frac{x-c_i}{\lambda}\exp\left(\frac{\beta}{\lambda}(a_i - x)\right) \underset{x\to\infty}{\longrightarrow} 0,$$

and

$$1-(x-c_i)\frac{h_i''(x)}{-h_i'(x)} + (1-\beta)(x-c_i)\frac{-h_i'(x)}{h_i(x)} = 1 - \frac{\beta}{\lambda}(x-c_i) \underset{x\to\infty}{\longrightarrow} -\infty.$$

Hence, we also have that $\tilde{G}'(x) < 0$ for high enough x.

Next, assume instead that $n \in f'$. Under NCES demand, the term in parenthesis in equation (ii) tends to

$$1-\sigma+(1-\beta)\tilde{\alpha}\tilde{\mu}^f + \beta\tilde{\alpha}\tilde{\mu}^f \frac{\sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^\beta}{H^{0\prime} + \sum_{l\in f'}\left(\sum_{j\in l\cap\mathcal{N}^{f'}} h_j(\hat{p}_j)\right)^\beta},$$

which, using equation (i), simplifies to

$$1-\sigma+(1-\beta)\tilde{\alpha}\tilde{\mu}^f + \tilde{\mu}^f(1-\tilde{\alpha}(1-\beta)) - 1 = -\sigma + \tilde{\mu}^f < 0,$$

implying that $\tilde{G}'(x) < 0$ for x high enough.

Under NMNL demand, the term in parenthesis in equation (ii) tends again to $-\infty$, so that $\tilde{G}'(x) < 0$ for x high enough.

It follows that $\tilde{G}$ is strictly decreasing over some interval (x^0, ∞). Therefore, $\tilde{G}(x^0) > \lim_{x\to\infty}\tilde{G}(x) = G(\hat{p})$, and $\hat{p}$ does not maximize G, a contradiction. Hence, $\hat{p} \in \prod_{j\in\mathcal{N}^f}[c_j, \infty)$ maximizes G, which concludes Step 3.

Step 4: There exists a unique price vector satisfying the first-order optimality conditions. The analysis in Section 2.3 implies that the price vector $\hat{p} \in \prod_{j\in\mathcal{N}^f}[c_j,\infty)$ satisfies the first-order conditions if and only if there exists a $\tilde{\mu}^f$ that is such that for every $i \in \mathcal{N}^f$, $\hat{p}_i = r_i(\tilde{\mu}^f)$, where

$$r_i(x) \equiv \begin{cases} \frac{\sigma}{\sigma - x}c_i & \text{in the case of NCES,} \\ \lambda x + c_i & \text{in the case of NMNL,} \end{cases}$$

and that satisfies

$$\tilde{\mu}^f\left(1-\tilde{\alpha}(1-\beta)\right) = 1 + \tilde{\alpha}\beta\tilde{\mu}^f \frac{\sum_{l\in f}\left(\sum_{j\in f} h_j(r_j(\tilde{\mu}^f))\right)^\beta}{H^{0\prime} + \sum_{l\in f}\left(\sum_{j\in f} h_j(r_j(\tilde{\mu}^f))\right)^\beta},$$

or, equivalently,

$$\tilde{\mu}^f(1-\tilde{\alpha}) = 1 - \tilde{\alpha}\beta\tilde{\mu}^f \frac{H^{0\prime}}{H^{0\prime} + \sum_{l\in f}\left(\sum_{j\in f} h_j(r_j(\tilde{\mu}^f))\right)^\beta}. \tag{iv}$$

As the left-hand side of equation (iv) is strictly increasing, whereas the right-hand side is strictly decreasing, that equation has at most one solution. By Step 3, that equation has a solution. Hence, there exists a unique price vector satisfying the first-order conditions. □

II Technical Results on Fitting-In Functions

The following results are proved in Nocke and Schutz (2018):

Lemma I. *The following holds for every $\alpha \in (0,1]$:*

(a) For every $x > 0$,

$$S'(x) = \frac{1}{x}\frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}. \tag{v}$$

(b) The elasticity of S, $\varepsilon(x) = xS'(x)/S(x)$, is strictly decreasing in x.

(c) S is strictly concave.

Proof. See Section XIII.3 in the Online Appendix to Nocke and Schutz (2018). □

We also require the following lemma:

Lemma II. *The continuous extension of S to $\mathbb{R}_+$ is $\mathcal{C}^3$. Moreover, $S(0) = 0$,*

$$S'(0) = \begin{cases} \alpha^{\frac{\alpha}{1-\alpha}} & \text{under NCES demand,} \\ e^{-1} & \text{under NMNL demand,} \end{cases}$$

$S''(0) = -2\alpha S'(0)^2$, and $S'''(0) = -3\alpha(1-2\alpha)S'(0)^3$.

The inverse function $\Theta \equiv S^{-1}$ is $\mathcal{C}^3$ on $[0,1)$. Moreover, $\Theta(0) = 0$, $\Theta'(0) = 1/S'(0)$, $\Theta''(0) = 2\alpha/S'(0)$, and $\Theta'''(0) = 3\alpha(1+2\alpha)/S'(0)$.

Proof. We start by computing $\lim_{x\downarrow 0}\frac{S(x)}{x}$. In the NMNL case,

$$\frac{S(x)}{x} = \mathrm{e}^{-m(x)} = \exp\left(\frac{-1}{1-S(x)}\right) \underset{x\downarrow 0}{\longrightarrow} \mathrm{e}^{-1}.$$

In the NCES case,

$$\frac{S(x)}{x} = (1-(1-\alpha)m(x))^{\frac{\alpha}{1-\alpha}} = \left(1 - \frac{1-\alpha}{1-\alpha S(x)}\right)^{\frac{\alpha}{1-\alpha}} \underset{x\downarrow 0}{\longrightarrow} \alpha^{\frac{\alpha}{1-\alpha}}.$$

Differentiating equation (v), we obtain

$$S''(x) = -\left(\frac{S(x)}{x}\right)^2 \frac{\alpha(2-S(x))(1-S(x))(1-\alpha S(x))}{(1-S(x)+\alpha S(x)^2)^3}. \tag{vi}$$

Differentiating once more gives

$$S'''(x) = -\left(\frac{S(x)}{x}\right)^3 \frac{\alpha(1-S(x))(1-\alpha S(x))}{(1-S(x)+\alpha S(x)^2)^5}\Big(3(1-2\alpha) - 4(1+\alpha)S(x) \\ + (1+13\alpha+6\alpha^2)S(x)^2 - 2\alpha(2+5\alpha)S(x)^3 + 3\alpha^2 S(x)^4\Big). \tag{vii}$$

Taking limits in equations (vi) and (vii) gives us the values of $S''(0)$ and $S'''(0)$.

Since S is $\mathcal{C}^3$ with strictly positive derivative on $\mathbb{R}_+$, that function establishes a $\mathcal{C}^3$-diffeomorphism from $\mathbb{R}_+$ to

$$\Big[S(0), \lim_{x\to\infty} S(x)\Big) = [0,1).$$

It follows that Θ is $\mathcal{C}^3$. Moreover,

$$\begin{aligned}
\Theta'(s) &= \frac{1}{S'\circ S^{-1}(s)},\\
\Theta''(s) &= -\frac{S''\circ S^{-1}(s)}{\left(S'\circ S^{-1}(s)\right)^3},\\
\Theta'''(s) &= -\frac{\frac{S'''\circ S^{-1}(s)}{S'\circ S^{-1}(s)}\left(S'\circ S^{-1}(s)\right)^3 - S''\circ S^{-1}(s)\times 3\left(S'\circ S^{-1}(s)\right)^2\frac{S''\circ S^{-1}(s)}{S'\circ S^{-1}(s)}}{\left(S'\circ S^{-1}(s)\right)^6},\\
&= \frac{1}{S'\circ S^{-1}(s)}\left(-\frac{S'''\circ S^{-1}(s)}{\left(S'\circ S^{-1}(s)\right)^3} + 3\left(\frac{S''\circ S^{-1}(s)}{\left(S'\circ S^{-1}(s)\right)^2}\right)^2\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Theta'(0) &= \frac{1}{S'(0)},\\
\Theta''(0) &= -\frac{1}{S'(0)}\frac{S''(0)}{S'(0)^2} = \frac{2\alpha}{S'(0)},\\
\Theta'''(0) &= \frac{1}{S'(0)}\left(-\frac{S'''(0)}{S'(0)^3} + 3\left(\frac{S''(0)}{S'(0)^2}\right)^2\right),\\
&= \frac{1}{S'(0)}\left(3\alpha(1-2\alpha) + 3(2\alpha)^2\right),\\
&= \frac{3\alpha(1+2\alpha)}{S'(0)}.
\end{aligned}$$

□

III Approximation Results Around Small Market Shares

III.1 Proof of Proposition 3

We prove a series of lemmas that jointly imply Proposition 3 as well as the third-order approximation mentioned in footnote 23.

We first approximate consumer surplus under oligopoly:

Lemma III. *$H^*(s) = \frac{H^0}{1-\sum_{g\in\mathcal{F}} s^g}$. Moreover, in the neighborhood of $s = 0$,*

$$CS(s) = \log H^0 + \sum_{f\in\mathcal{F}} s^f + \frac{1}{2}\left(\sum_{f\in\mathcal{F}} s^f\right)^2 + \frac{1}{3}\left(\sum_{f\in\mathcal{F}} s^f\right)^3 + o\left(\|s\|^3\right).$$

Proof. The first part of the lemma follows immediately from the equilibrium condition

$$\frac{H^0}{H^*} + \sum_{g\in\mathcal{F}} s^g = 1.$$

The second part of the lemma follows from the fact that, in the neighborhood of $x = 0$,

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3). \qquad \square$$

Next, we compute the first, second, and third (cross-)partial derivatives of the type vector $T(s)$:

Lemma IV. *For every $(f, f') \in \mathcal{F}^2$,*

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{otherwise.} \end{cases}$$

For every $(f, f', f'') \in \mathcal{F}^3$,

$$\left.\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} 2(1+\alpha) & \text{if } f = f' = f'', \\ 0 & \text{if } f' \neq f \text{ and } f'' \neq f, \\ \frac{H^0}{S'(0)} & \text{otherwise.} \end{cases}$$

Finally, for every $(f, f', f'', f''') \in \mathcal{F}^4$,

$$\left.\frac{\partial^3 T^f}{\partial s^{f'} \partial s^{f''} \partial s^{f'''}}\right|_{s=0} = \frac{H^0}{S'(0)} \begin{cases} 6 + 9\alpha + 6\alpha^2 & \text{if } f = f' = f'' = f''', \\ 2\alpha + 4 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f), \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{P}^1(f) = \left\{(f^1, f^2, f^3) \in \mathcal{F}^3 :\ f = f^i \neq f^j, f^k,\ \textit{for some permutation } (i,j,k) \textit{ of } (1,2,3)\right\},$$

and

$$\mathcal{P}^2(f) = \left\{(f^1, f^2, f^3) \in \mathcal{F}^3 :\ f = f^i = f^j \neq f^k,\ \textit{for some permutation } (i,j,k) \textit{ of } (1,2,3)\right\}.$$

Proof. Let $f \in \mathcal{F}$. Since $s^f = S\left(\frac{T^f}{H^*(s)}\right)$, we have that

$$T^f = H^* S^{-1}(s^f) = H^0 \frac{\Theta(s^f)}{1 - \sum_{g\in\mathcal{F}} s^g} \equiv H^0 \Theta(s^f)\Psi(s),$$

where we have used the inverse function Θ that was defined in Lemma II.

Note that, for every $(f, f', f'') \in \mathcal{F}^3$,

$$\begin{aligned}
\Psi(0) = \left.\frac{\partial \Psi}{\partial s^f}\right|_{s=0} &= 1, \\
\left.\frac{\partial^2 \Psi}{\partial s^f \partial s^{f'}}\right|_{s=0} &= 2, \\
\left.\frac{\partial^3 \Psi}{\partial s^f \partial s^{f'} \partial s^{f''}}\right|_{s=0} &= 6.
\end{aligned}$$

Therefore, for every $(f, f') \in \mathcal{F}^2$,

$$\left.\frac{\partial T^f}{\partial s^{f'}}\right|_{s=0} = H^0 \left.\left(\frac{\partial \Theta(s^f)}{\partial s^{f'}}\Psi(s) + \Theta(s^f)\frac{\partial \Psi}{\partial s^{f'}}\right)\right|_{s=0} = \begin{cases} \frac{H^0}{S'(0)} & \text{if } f = f', \\ 0 & \text{if } f \neq f'. \end{cases}$$

For every $(f, f', f'') \in \mathcal{F}^3$,

$$\begin{aligned}
\left.\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= H^0 \left.\left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}}\Psi(s) + \Theta(s^f)\frac{\partial^2 \Psi}{\partial s^{f'} \partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}}\frac{\partial \Psi}{\partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}}\frac{\partial \Psi}{\partial s^{f'}}\right)\right|_{s=0}, \\
&= H^0 \left.\left(\frac{\partial^2 \Theta(s^f)}{\partial s^{f'} \partial s^{f''}} + \frac{\partial \Theta(s^f)}{\partial s^{f'}} + \frac{\partial \Theta(s^f)}{\partial s^{f''}}\right)\right|_{s=0}, \\
&= H^0 \times \begin{cases} \Theta''(0) + 2\Theta'(0) & \text{if } f = f' = f'', \\ 0 & \text{if } f', f'' \neq f, \\ \Theta'(0) & \text{otherwise,} \end{cases} \\
&= \frac{H^0}{S'(0)} \times \begin{cases} 2(\alpha + 1) & \text{if } f = f' = f'', \\ 0 & \text{if } f', f'' \neq f, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

Finally, for every $(f, f', f'', f''') \in \mathcal{F}^3$,

$$\begin{aligned}
\left.\frac{\partial^3 T^f}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}}\right|_{s=0} &= H^0 \left(\frac{\partial^3\Theta(s^f)}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}}\Psi(s) + \Theta(s^f)\frac{\partial^3\Psi}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}} + \frac{\partial^2\Theta(s^f)}{\partial s^{f'}\partial s^{f''}}\frac{\partial\Psi}{\partial s^{f'''}}\right. \\
&\quad + \frac{\partial\Theta(s^f)}{\partial s^{f'''}}\frac{\partial^2\Psi}{\partial s^{f'}\partial s^{f''}} + \frac{\partial^2\Theta(s^f)}{\partial s^{f'}\partial s^{f'''}}\frac{\partial\Psi}{\partial s^{f''}} + \frac{\partial\Theta(s^f)}{\partial s^{f'}}\frac{\partial^2\Psi}{\partial s^{f''}\partial s^{f'''}} \\
&\quad \left.\left. + \frac{\partial^2\Theta(s^f)}{\partial s^{f''}\partial s^{f'''}}\frac{\partial\Psi}{\partial s^{f'}} + \frac{\partial\Theta(s^f)}{\partial s^{f''}}\frac{\partial^2\Psi}{\partial s^{f'}\partial s^{f'''}}\right)\right|_{s=0}, \\
&= H^0 \left(\frac{\partial^3\Theta(s^f)}{\partial s^{f'}\partial s^{f''}\partial s^{f'''}} + \frac{\partial^2\Theta(s^f)}{\partial s^{f'}\partial s^{f''}} + 2\frac{\partial\Theta(s^f)}{\partial s^{f'''}} + \frac{\partial^2\Theta(s^f)}{\partial s^{f'}\partial s^{f'''}}\right. \\
&\quad \left.\left. + 2\frac{\partial\Theta(s^f)}{\partial s^{f'}} + \frac{\partial^2\Theta(s^f)}{\partial s^{f''}\partial s^{f'''}} + 2\frac{\partial\Theta(s^f)}{\partial s^{f''}}\right)\right|_{s=0}, \\
&= \frac{H^0}{S'(0)}\begin{cases} 3\alpha(1+2\alpha) + 3(2\alpha+2) & \text{if } f = f' = f'' = f''', \\ 2\alpha + 2 + 2 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f), \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f), \\ 0 & \text{otherwise,} \end{cases} \\
&= \frac{H^0}{S'(0)}\begin{cases} 6 + 9\alpha + 6\alpha^2 & \text{if } f = f' = f'' = f''', \\ 2\alpha + 4 & \text{if } (f', f'', f''') \in \mathcal{P}^2(f), \\ 2 & \text{if } (f', f'', f''') \in \mathcal{P}^1(f), \\ 0 & \text{otherwise.} \end{cases} \qquad \square
\end{aligned}$$

To ease notation, let $\bar{s} = \sum_{g\in\mathcal{F}} s^g$. We now use Lemma IV to obtain a third-order Taylor approximation of $T^f(s)$ in the neighborhood of $s = 0$:

Lemma V. *In the neighborhood of $s = 0$,*

$$T^f(s) = \frac{H^0}{S'(0)}\left(s^f + \left(\alpha(s^f)^2 + s^f\bar{s}\right) + \left(\frac{\alpha(1+2\alpha)}{2}(s^f)^3 + \alpha(s^f)^2\bar{s} + s^f\bar{s}^2\right)\right) + o(\|s\|^3).$$

Proof. By Lemma IV, first-order terms are simply given by $\frac{H^0}{S'(0)}s^f$. Second-order terms are given by

$$\frac{H^0}{S'(0)}\frac{1}{2}\left(2(1+\alpha)(s^f)^2 + 2s^f\sum_{g\neq f}s^g\right) = \frac{H^0}{S'(0)}\left(\alpha(s^f)^2 + s^f\bar{s}\right).$$

Finally, third-order terms are:

$$\begin{aligned}
&\frac{H^0}{S'(0)}\frac{1}{6}\left((6+9\alpha+6\alpha^2)(s^f)^3 + (2\alpha+4)\sum_{(f',f'',f''')\in\mathcal{P}^2(f)} s^{f'}s^{f''}s^{f'''} + 2\sum_{(f',f'',f''')\in\mathcal{P}^1(f)} s^{f'}s^{f''}s^{f'''}\right), \\
&= \frac{H^0}{S'(0)}\frac{1}{6}\left((6+9\alpha+6\alpha^2)(s^f)^3 + 3(2\alpha+4)(s^f)^2\sum_{g\neq f}s^g + 6s^f\sum_{g,g'\neq f}s^g s^{g'}\right),
\end{aligned}$$

$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((6 + 9\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2(\bar{s} - s^f) + 6s^f(\bar{s} - s^f)^2 \right),$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((-6 + 3\alpha + 6\alpha^2)(s^f)^3 + 3(2\alpha + 4)(s^f)^2\bar{s} + 6s^f(\bar{s}^2 - 2\bar{s}s^f + (s^f)^2) \right),$$
$$= \frac{H^0}{S'(0)} \frac{1}{6} \left((3\alpha + 6\alpha^2)(s^f)^3 + 6\alpha(s^f)^2\bar{s} + 6s^f\bar{s}^2 \right),$$

The lemma follows by Taylor's theorem. □

We recall the definition of the dispersion measure $\Gamma(s)$:

$$\Gamma(s) = \sum_{f \in \mathcal{F}} (s^f)^3.$$

The following lemma gives a third-order Taylor approximation of the sum of the types:

Lemma VI. *In the neighborhood of $s = 0$,*

$$\sum_{f \in \mathcal{F}} \frac{S'(0)}{H^0} T^f(s) = \bar{s} + \left(\alpha\, HHI(s) + \bar{s}^2 \right) + \left(\frac{\alpha(1 + 2\alpha)}{2} \Gamma(s) + \alpha\, HHI(s)\bar{s} + \bar{s}^3 \right) + o(\|s\|^3).$$

Proof. Immediate. □

Let $\mathrm{CS}^m(s)$ be consumer surplus under monopolistic competition. Recall that all the firms set their normalized markups equal to 1 under monopolistic competition. Hence, in the case of NMNL demand,

$$\mathrm{CS}^m(s) = \log \left(H^0 + \sum_{f \in \mathcal{F}} T^f(s)\, \mathrm{e}^{-1} \right) = \log H^0 + \log \left(1 + \sum_{f \in \mathcal{F}} T^f(s) \frac{S'(0)}{H^0} \right). \qquad \text{(viii)}$$

Similarly, in the case of NCES demand,

$$\mathrm{CS}^m(s) = \log \left(H^0 + \sum_{f \in \mathcal{F}} T^f(s) \alpha^{\frac{\alpha}{1-\alpha}} \right) = \log H^0 + \log \left(1 + \sum_{f \in \mathcal{F}} T^f(s) \frac{S'(0)}{H^0} \right). \qquad \text{(ix)}$$

We now provide a third-order Taylor expansion of $\mathrm{CS}^m(s)$:

Lemma VII. *In the neighborhood of $s = 0$,*

$$CS^m(s) = \log H^0 + \bar{s} + \frac{1}{2}\bar{s}^2 + \frac{1}{3}\bar{s}^3 + \alpha\, HHI(s) + \frac{\alpha(1 + 2\alpha)}{2} \Gamma(s) + o(\|s\|^3).$$

Proof. Recall that, at the third order in the neighborhood of $x = 0$,

$$\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3).$$

Combining this with Lemma VI, and eliminating higher-order terms, we obtain

$$\begin{aligned}\mathrm{CS}^m(s) = &\ \log H^0 + \bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\,\mathrm{HHI}(s)\bar{s} + \bar{s}^3 \\ &- \frac{1}{2}\left(\bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2\right)^2 + \frac{1}{3}\bar{s}^3 + o(\|s\|^3), \\ = &\ \log H^0 + \bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\,\mathrm{HHI}(s)\bar{s} + \bar{s}^3 \\ &- \frac{1}{2}\left(\bar{s}^2 + 2\alpha\,\mathrm{HHI}(s)\bar{s} + 2\bar{s}^3\right) + \frac{1}{3}\bar{s}^3 + o(\|s\|^3), \\ = &\ \log H^0 + \bar{s} + \frac{1}{2}\bar{s}^2 + \frac{1}{3}\bar{s}^3 + \alpha\,\mathrm{HHI}(s) + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + o(\|s\|^3). \qquad \square\end{aligned}$$

Combining Lemmas III and VII, we obtain the approximation results for the distortion to consumer surplus that were announced in Proposition 3 and footnote 23:

Lemma VIII. *In the neighborhood of $s = 0$,*

$$CS(s) - CS^m(s) = -\alpha\, HHI(s) - \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + o\left(\|s\|^3\right).$$

Next, we turn our attention to profits. Let

$$\Pi(s) = \sum_{f\in\mathcal{F}}\left(\frac{1}{1-\alpha s^f} - 1\right)$$

be aggregate profit.

Lemma IX. *In the neighborhood of $s = 0$,*

$$\Pi(s) = \alpha\bar{s} + \alpha^2\, HHI(s) + \alpha^3\Gamma(s) + o\left(\|s\|^3\right).$$

Proof. This follows immediately from the fact that, in the neighborhood of $x = 0$,

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + o(x^3). \qquad \square$$

Let $\Pi^m(s)$ be aggregate profit under monopolistic competition. Under NMNL demand,

$$\Pi^m(s) = \sum_{f\in\mathcal{F}}\frac{T^f(s)\,\mathrm{e}^{-1}}{H^0 + \sum_{g\in\mathcal{F}} T^g(s)\,\mathrm{e}^{-1}} = 1 - \frac{1}{1 + \sum_{g\in\mathcal{F}} T^g(s)\frac{S'(0)}{H^0}}.$$

Under NCES demand,

$$\Pi^m(s) = \sum_{f\in\mathcal{F}}\alpha\frac{T^f(s)\alpha^{\frac{\alpha}{1-\alpha}}}{H^0 + \sum_{g\in\mathcal{F}} T^g(s)\alpha^{\frac{\alpha}{1-\alpha}}} = \alpha\left(1 - \frac{1}{1 + \sum_{g\in\mathcal{F}} T^g(s)\frac{S'(0)}{H^0}}\right).$$

Lemma X. *In the neighborhood of $s = 0$,*

$$\Pi^m(s) = \alpha\left(\bar{s} + \alpha\, HHI(s) + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) - \alpha\, HHI(s)\bar{s}\right) + o(\|s\|^3).$$

Proof. Note that, at the third order in the neighborhood of $x = 0$,

$$1 - \frac{1}{1+x} = x - x^2 + x^3 + o(x^3).$$

Combining this with the definition of Π^m and Lemma VI, and eliminating higher-order terms, we obtain:

$$\begin{aligned}
\Pi^m(s) &= \alpha\left(\bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\,\mathrm{HHI}(s)\bar{s} + \bar{s}^3\right.\\
&\qquad \left. - \left(\bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2\right)^2 + \bar{s}^3\right) + o(\|s\|^3),\\
&= \alpha\left(\bar{s} + \alpha\,\mathrm{HHI}(s) + \bar{s}^2 + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) + \alpha\,\mathrm{HHI}(s)\bar{s} + \bar{s}^3\right.\\
&\qquad \left. - \left(\bar{s}^2 + 2\alpha\,\mathrm{HHI}(s)\bar{s} + 2\bar{s}^3\right) + \bar{s}^3\right) + o(\|s\|^3),\\
&= \alpha\left(\bar{s} + \alpha\,\mathrm{HHI}(s) + \frac{\alpha(1+2\alpha)}{2}\Gamma(s) - \alpha\,\mathrm{HHI}(s)\bar{s}\right) + o(\|s\|^3).
\end{aligned}$$

□

Combining Lemmas VIII, IX, and X delivers the approximation of the aggregate surplus distortion announced in Proposition 3 and footnote 23:

Lemma XI. *In the neighborhood of $s = 0$,*

$$AS(s) - AS^m(s) = -\alpha\left(HHI(s)(1-\alpha\bar{s}) + \frac{1}{2}(1+3\alpha)\Gamma(s)\right) + o\left(\|s\|^3\right).$$

III.2 Proof of Proposition 5

We prove a series of lemmas that jointly imply Proposition 5.

Recall from Appendix III.1 that $H^*(s)$ is the equilibrium value of the aggregator given the vector of market shares s. To ease notation, let $\overline{H}(s) \equiv H^*(\bar{s}(s))$ be the post-merger equilibrium value of the aggregator. We first provide an approximation of the market power effect of the merger, measured in terms of consumer surplus—the first part of Proposition 5:

Lemma XII. *In the neighborhood of $s = 0$,*

$$CS(\bar{s}(s)) - CS(s) = -\alpha\Delta^M\, HHI(s) + o(\|s\|^2).$$

Proof. By definition of $\overline{H}$, we have that

$$\frac{H^0}{\overline{H}} + \sum_{g\in\overline{\mathcal{F}}} S\left(\frac{T^g}{\overline{H}}\right) = 1.$$

Totally differentiating this expression, we obtain:

$$-\frac{d\overline{H}}{\overline{H}}\left(\frac{H^0}{\overline{H}} + \sum_{g\in\overline{\mathcal{F}}} \frac{T^g}{\overline{H}} S'\left(\frac{T^g}{\overline{H}}\right)\right) + \frac{1}{\overline{H}}\sum_{g\in\overline{\mathcal{F}}} S'\left(\frac{T^g}{\overline{H}}\right)\sum_{f\in\mathcal{F}}\frac{\partial T^g}{\partial s^f} ds^f = 0.$$

Hence,

$$\frac{\partial \overline{H}}{\partial s^f} = \overline{H}\frac{\sum_{g\in\overline{\mathcal{F}}} S'\left(\frac{T^g}{\overline{H}}\right)\frac{\partial T^g}{\partial s^f}}{H^0 + \sum_{g\in\overline{\mathcal{F}}} T^g S'\left(\frac{T^g}{\overline{H}}\right)}.$$

Hence, by Lemma V and since $T^M = \sum_{g\in\mathcal{M}} T^g$,

$$\left.\frac{\partial \overline{H}}{\partial s^f}\right|_{s=0} = H^0.$$

Next, we compute the Hessian of $\overline{H}$. Note that, for every $f, f' \in \mathcal{F}$

$$\begin{aligned}
\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} &= \frac{\partial \overline{H}}{\partial s^{f'}} \times 1 + H^0 \times \frac{1}{(H^0)^2}\left(\left(\sum_{g\in\overline{\mathcal{F}}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right)\right) H^0\right. \\
&\quad \left. - H^0\left(\sum_{g\in\overline{\mathcal{F}}}\frac{\partial T^g}{\partial s^f} S'(0)\right)\right), \\
&= H^0 + \sum_{g\in\overline{\mathcal{F}}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right) - \sum_{g\in\overline{\mathcal{F}}}\frac{\partial T^g}{\partial s^f} S'(0), \\
&= \sum_{g\in\overline{\mathcal{F}}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right), \\
&= \left(\frac{\partial^2 T^M}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^M}{\partial s^f}\frac{\partial T^M}{\partial s^{f'}} S''(0)\right) \\
&\quad + \sum_{g\in\mathcal{O}}\left(\frac{\partial^2 T^g}{\partial s^f \partial s^{f'}} S'(0) + \frac{1}{H^0}\frac{\partial T^g}{\partial s^f}\frac{\partial T^g}{\partial s^{f'}} S''(0)\right).
\end{aligned}$$

Assume first that $f \in \mathcal{O}$ and/or $f' \in \mathcal{O}$. Then, by Lemma V and since $T^M = \sum_{g\in\mathcal{M}} T^g$,

$$\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} = \begin{cases} 2H^0 & \text{if } f \neq f', \\ \frac{H^0}{S'(0)} 2(1+\alpha) S'(0) + \frac{1}{H^0}\left(\frac{H^0}{S'(0)}\right)^2 S''(0) & \text{if } f = f', \end{cases}$$

$$= 2H^0.$$

Next, assume instead that $f, f' \in \mathcal{M}$. Then,

$$\left.\frac{\partial^2 \overline{H}}{\partial s^f \partial s^{f'}}\right|_{s=0} = \begin{cases} 2H^0 + \frac{1}{H^0}\left(\frac{H^0}{S'(0)}\right)^2 S''(0) & \text{if } f \neq f', \\ \frac{H^0}{S'(0)} 2(1+\alpha)S'(0) + \frac{1}{H^0}\left(\frac{H^0}{S'(0)}\right)^2 S''(0) & \text{if } f = f', \end{cases}$$
$$= \begin{cases} 2H^0(1-\alpha) & \text{if } f \neq f', \\ 2H^0 & \text{if } f = f'. \end{cases}$$

By Taylor's theorem,

$$\overline{H}(s) = H^0 + H^0 \sum_{f\in\mathcal{F}} s^f + \frac{H^0}{2}\left(2\sum_{f,g\in\mathcal{F}} s^f s^g - 2\alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g\right) + o(\|s\|^2),$$
$$= H^0\left(1 + \sum_{f\in\mathcal{F}} s^f + \left(\sum_{f\in\mathcal{F}} s^f\right)^2 - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g\right) + o(\|s\|^2).$$

Using the fact that $\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$ in the neighborhood of $x = 0$, this implies that

$$\log \overline{H}(s) = \log H^0 + \sum_{f\in\mathcal{F}} s^f + \left(\sum_{f\in\mathcal{F}} s^f\right)^2 - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g - \frac{1}{2}\left(\sum_{f\in\mathcal{F}} s^f\right)^2 + o(\|s\|^2),$$
$$= \log H^*(s) - \alpha \sum_{\substack{f,g\in\mathcal{M}\\ f\neq g}} s^f s^g + o(\|s\|^2), \text{ by Lemma III,}$$
$$= \log H^*(s) - \alpha \Delta^M \,\mathrm{HHI}(s) + o(\|s\|^2). \qquad \square$$

Next, we approximate post-merger market shares:

Lemma XIII. *In the neighborhood of $s = 0$, for every $f \in \mathcal{O}$*

$$\bar{s}^f = s^f + o(\|s\|^2),$$

and

$$\bar{s}^M = \sum_{f\in\mathcal{M}} s^f - \alpha\Delta^M \, HHI(s) + o(\|s\|^2).$$

Proof. By definition, for every $f \in \overline{\mathcal{F}}$,

$$\bar{s}^f = S\left(\frac{T^f}{\overline{H}}\right).$$

For every $f \in \overline{\mathcal{F}}$ and $f' \in \mathcal{F}$,

$$\frac{\partial \bar{s}^f}{\partial s^{f'}} = \frac{1}{\overline{H}}\left(\frac{\partial T^f}{\partial s^{f'}} - \frac{T^f}{\overline{H}}\frac{\partial \overline{H}}{\partial s^{f'}}\right) S'\left(\frac{T^f}{\overline{H}}\right).$$

It follows that

$$\left.\frac{\partial \bar{s}^f}{\partial s^{f'}}\right|_{s=0} = \begin{cases} 0 & \text{if } f \neq f' \text{ and } (f \neq M \text{ or } f' \notin \mathcal{M}), \\ 1 & \text{otherwise.} \end{cases}$$

For every $f \in \overline{\mathcal{F}}$ and $f', f'' \in \mathcal{F}$,

$$\begin{aligned} \left.\frac{\partial^2 \bar{s}^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= -\frac{\partial \overline{H}}{\partial s^{f''}} \frac{1}{\overline{H}^2} \frac{\partial T^f}{\partial s^{f'}} S'(0) + \frac{1}{\overline{H}}\left(\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}} - \frac{1}{\overline{H}} \frac{\partial T^f}{\partial s^{f''}} \frac{\partial \overline{H}}{\partial s^{f'}}\right) S'(0) \\ &\quad + \frac{1}{\overline{H}^2} \frac{\partial T^f}{\partial s^{f'}} \frac{\partial T^f}{\partial s^{f''}} S''(0), \\ &= -\frac{1}{H^0} \frac{\partial T^f}{\partial s^{f'}} S'(0) + \frac{1}{H^0}\left(\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^f}{\partial s^{f''}}\right) S'(0) + \frac{1}{(H^0)^2} \frac{\partial T^f}{\partial s^{f'}} \frac{\partial T^f}{\partial s^{f''}} S''(0), \\ &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^f}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^f}{\partial s^{f'}} - \frac{\partial T^f}{\partial s^{f''}}\right) + \frac{S''(0)}{(H^0)^2} \frac{\partial T^f}{\partial s^{f'}} \frac{\partial T^f}{\partial s^{f''}}. \end{aligned}$$

Suppose first that $f \neq M$, so that $f \in \mathcal{F}$. Clearly, if $f' \neq f$ and $f'' \neq f$, then,

$$\left.\frac{\partial^2 \bar{s}^f}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = 0.$$

If $f'' \neq f$, then

$$\left.\frac{\partial^2 \bar{s}^f}{\partial s^{f} \partial s^{f''}}\right|_{s=0} = \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^f}{\partial s^{f} \partial s^{f''}} - \frac{\partial T^f}{\partial s^{f}}\right) = 0.$$

Finally,

$$\begin{aligned} \left.\frac{\partial^2 \bar{s}^f}{\partial (s^{f})^2}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^f}{\partial (s^f)^2} - 2\frac{\partial T^f}{\partial s^{f}}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{\partial T^f}{\partial s^f}\right)^2, \\ &= \frac{S'(0)}{H^0}\left(\frac{H^0}{S'(0)} 2(1+\alpha) - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2, \\ &= 0. \end{aligned}$$

Next, assume that $f = M$. Clearly, if $f', f'' \notin \mathcal{M}$, then

$$\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} = 0.$$

Next assume that $f'' \notin \mathcal{M}$ and $f' \in \mathcal{M}$. Then,

$$\begin{aligned}\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^M}{\partial s^{f'}}\right), \\ &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^{f'}}{\partial s^{f'}}\right), \\ &= 0.\end{aligned}$$

Next, assume that $f', f'' \in \mathcal{M}$. Then,

$$\begin{aligned}\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'} \partial s^{f''}} - \frac{\partial T^{f'}}{\partial s^{f'}} - \frac{\partial T^{f''}}{\partial s^{f''}}\right) + \frac{S''(0)}{(H^0)^2}\frac{\partial T^{f'}}{\partial s^{f'}}\frac{\partial T^{f''}}{\partial s^{f''}}, \\ &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^M}{\partial s^{f'} \partial s^{f''}} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2.\end{aligned}$$

Hence, if $f' = f''$, then

$$\begin{aligned}\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial (s^{f'})^2} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2, \\ &= 0.\end{aligned}$$

If instead $f' \neq f''$, then

$$\begin{aligned}\left.\frac{\partial^2 \bar{s}^M}{\partial s^{f'} \partial s^{f''}}\right|_{s=0} &= \frac{S'(0)}{H^0}\left(\frac{\partial^2 T^{f'}}{\partial s^{f'} \partial s^{f''}} + \frac{\partial^2 T^{f''}}{\partial s^{f'} \partial s^{f''}} - 2\frac{H^0}{S'(0)}\right) + \frac{S''(0)}{(H^0)^2}\left(\frac{H^0}{S'(0)}\right)^2, \\ &= -2\alpha.\end{aligned}$$

The lemma follows by Taylor's theorem. □

Let

$$\begin{aligned}\Pi(s) &= \sum_{f \in \mathcal{F}}\left(\frac{1}{1-\alpha s^f} - 1\right), \\ \text{and } \overline{\Pi}(s) &= \sum_{f \in \overline{\mathcal{F}}}\left(\frac{1}{1-\alpha \bar{s}^f} - 1\right),\end{aligned}$$

be aggregate profits, pre- and post-merger, respectively.

Lemma XIV. *In the neighborhood of $s = 0$,*

$$\overline{\Pi}(s) - \Pi(s) = o(\|s\|^2).$$

Proof. By Lemma XIII, and since $\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + o(\|x\|^2)$ in the neighborhood of $x = 0$, we have that

$$\Pi(s) = \alpha \sum_{f\in\mathcal{F}} s^f + \alpha^2 \sum_{f\in\mathcal{F}} (s^f)^2 + o(\|s\|^2),$$

and

$$\begin{aligned}
\overline{\Pi}(s) &= \frac{1}{1-\alpha \bar{s}^M} - 1 + \sum_{f\in\mathcal{O}} \left(\frac{1}{1-\alpha \bar{s}^f} - 1 \right), \\
&= \alpha \left(\sum_{f\in\mathcal{M}} s^f - \alpha \sum_{\substack{f,g\in\mathcal{M} \\ f\neq g}} s^f s^g \right) + \alpha^2 \left(\sum_{f\in\mathcal{M}} s^f \right)^2 + \alpha \sum_{f\in\mathcal{O}} s^f + \alpha^2 \sum_{f\in\mathcal{O}} (s^f)^2 + o(\|s\|^2), \\
&= \alpha \sum_{f\in\mathcal{F}} s^f + \alpha^2 \sum_{f\in\mathcal{F}} (s^f)^2 + o(\|s\|^2), \\
&= \Pi(s) + o(\|s\|^2).
\end{aligned}$$

□

Combining Lemmas XII and XIV proves the second part of Proposition 5:

Lemma XV. *In the neighborhood of $s = 0$,*

$$AS(\bar{s}(s)) - AS(s) = -\alpha \Delta^M \, HHI(s) + o(\|s\|^2).$$

IV Approximation Results Around Monopolistic Competition Conduct

This section is organized as follows. We first provide more details on our treatment of firm conduct in Section IV.1. We then prove Proposition 4 in Section IV.2, and Proposition 6 in Section IV.3.

IV.1 Firm Conduct

Let $\theta \in [0, 1]$ be a conduct parameter as defined at the end of Section 2.3. The first-order condition for product $i \in n \in f$ is given by

$$\frac{H_n^{\beta-1}}{H} \left(-h_i' - (p_i - c_i) h_i'' + (1-\beta) \frac{\partial H_n}{\partial p_i} \frac{\sum_{j\in n} (p_j - c_j) h_j'}{H_n} \right.$$

$$+ \theta \times \frac{H_n^{1-\beta}}{H} \frac{\partial H}{\partial p_i} \sum_{l \in f} H_l^{\beta-1} \sum_{j \in l} (p_j - c_j) h'_j \Bigg) = 0,$$

which can be rewritten as

$$\frac{p_i - c_i}{p_i} \frac{p_i h''_i}{-h'_i} = 1 + (1-\beta) \frac{\sum_{j \in n} (p_j - c_j)(-h'_j)}{H_n} + \theta\beta \frac{1}{H} \sum_{l \in f} H_l^{\beta-1} \sum_{j \in l} (p_j - c_j)(-h'_j), \qquad \text{(x)}$$

so that the common ι-markup property within nest n continues to hold. Let $\tilde{\mu}_n$ be firm f's ι-markup in nest n. Then, using equation (4), equation (x) simplifies to

$$\tilde{\mu}_n \left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \theta\tilde{\alpha}\beta \frac{1}{H} \sum_{l \in f} \tilde{\mu}^l H_l^\beta, \qquad \text{(xi)}$$

so that $\tilde{\mu}_n = \tilde{\mu}_{n'} \equiv \tilde{\mu}^f$ for every $n, n' \in f$. Using the common ι-markup property both within nest and across nests allows us to further simplify equation (xi):

$$\tilde{\mu}^f \left(1 - \tilde{\alpha}(1-\beta)\right) = 1 + \theta\tilde{\alpha}\beta\tilde{\mu}^f s^f.$$

Defining $\mu^f \equiv \tilde{\mu}^f \left(1 - \tilde{\alpha}(1-\beta)\right)$ as we did in Section 2.3, this implies that

$$\mu^f = \frac{1}{1 - \theta\alpha s^f}. \qquad \text{(xii)}$$

As the conduct parameter θ does not affect the demand system, it is still the case that

$$s^f = \begin{cases} \frac{T^f}{H} \left(1 - (1-\alpha)\mu^f\right)^{\frac{\alpha}{1-\alpha}} & \text{under NCES demand,} \\ \frac{T^f}{H} \mathrm{e}^{-\mu^f} & \text{under NMNL demand.} \end{cases} \qquad \text{(xiii)}$$

Thus, firm f's markup and market share jointly solve equations (xii) and (xiii). This pins down the fitting-in functions $m(T^f/H, \theta)$ and $S(T^f/H, \theta)$. The profit fitting-in function is given by

$$\begin{aligned} \pi(T^f/H, \theta) &= \frac{\beta}{H} \sum_{l \in f} H_l^{\beta-1} \sum_{j \in f} (p_j - c_j)(-h'_j), \\ &= \frac{\beta}{H} \tilde{\mu}^f \tilde{\alpha} \sum_{l \in f} H_l^\beta, \text{ using equation (4),} \\ &= \alpha\mu^f s^f, \text{ by definition of } \mu^f, \ s^f, \text{ and } \alpha, \\ &= \alpha m\left(\frac{T^f}{H}, \theta\right) S\left(\frac{T^f}{H}, \theta\right), \end{aligned}$$

$$= \frac{\alpha S\left(\frac{T^f}{H}, \theta\right)}{1 - \alpha\theta S\left(\frac{T^f}{H}, \theta\right)}.$$

The equilibrium aggregator level $H^*(\theta)$ uniquely solves the equation

$$\frac{H^0}{H} + \sum_{f \in \mathcal{F}} S\left(\frac{T^f}{H}, \theta\right) = 1.$$

It is easy to see that $H^*(\theta)$, $m(\cdot, \theta)$, $S(\cdot, \theta)$, and $\pi(\cdot, \theta)$ all tend to their value under monopolistic competition as θ tends to 0, and to their value under fully-fledged oligopoly as θ tends to 1, as stated at the end of Section 2.3.

IV.2 Proof of Proposition 4

We prove a series of lemmas that jointly imply Proposition 4.

Recall from Section IV.1 that the markup and market-share fitting-in functions, $m(x, \theta)$ and $S(x, \theta)$, jointly solve the system

$$\mu = \frac{1}{1 - \theta\alpha s},$$
$$s = \begin{cases} x\left(1 - (1-\alpha)\mu\right)^{\frac{\alpha}{1-\alpha}} & \text{under NCES demand,} \\ x\,\mathrm{e}^{-\mu} & \text{under NMNL demand.} \end{cases}$$

We compute the partial derivatives of $S(x, \theta)$ at $\theta = 0$:

Lemma XVI. *For every $\alpha \in (0, 1]$ and $x > 0$,*

$$\left.\frac{\partial S}{\partial x}\right|_{(x,0)} = \frac{S(x,0)}{x},$$
$$\text{and} \quad \left.\frac{\partial S}{\partial \theta}\right|_{(x,0)} = -\alpha S(x,0)^2.$$

Proof. Under NMNL demand,

$$S = x\,\mathrm{e}^{-m} = x \exp\left(-\frac{1}{1 - \theta S}\right).$$

Hence, at $\theta = 0$,

$$dS = \frac{S}{x}dx - S^2 d\theta,$$

which proves the lemma for the case $\alpha = 1$.

Under NCES demand,

$$S = x\,(1-(1-\alpha)m)^{\frac{\alpha}{1-\alpha}} = x\left(1-\frac{1-\alpha}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}} = x\left(\alpha\frac{1-\theta S}{1-\theta\alpha S}\right)^{\frac{\alpha}{1-\alpha}}.$$

Hence, at $\theta = 0$,

$$\begin{aligned} dS &= \frac{S}{x}dx + \frac{\alpha}{1-\alpha}S\frac{1-\alpha\theta S}{1-\theta S}\frac{1}{(1-\alpha\theta S)^2}\Big(\left(-\theta(1-\alpha\theta S)+\alpha\theta(1-\theta S)\right)dS \\ &\quad + \left(-S(1-\alpha\theta S)+\alpha S(1-\theta S)\right)d\theta\Big), \\ &= \frac{S}{x}dx - \alpha S^2, \end{aligned}$$

which proves the lemma for the case $\alpha < 1$. □

Fix a profile of types $(T^f)_{f\in\mathcal{F}}$ and a value of the outside option $H^0 \geq 0$. We compute $H^{*\prime}(0)$, and use this derivative to obtain the first part of Proposition 4:

Lemma XVII. *The following holds:*

$$\left.\frac{d\log H^*}{d\theta}\right|_{\theta=0} = -\alpha\, HHI(0).$$

This implies that, in the neighborhood of $\theta = 0$,

$$CS(\theta) - CS(0) = -\alpha\, HHI(\theta)\theta + o(\theta).$$

Proof. Recall that $H^*(\theta)$ is pinned down by the equilibrium condition

$$\frac{H^0}{H^*} + \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*},\theta\right) = 1.$$

Totally differentiating the equilibrium condition, we obtain:

$$-\frac{dH^*}{H^*}\left(\frac{H^0}{H^*} + \sum_{f\in\mathcal{F}}\frac{T^f}{H^*}\frac{\partial S}{\partial(T^f/H^*)}\left(\frac{T^f}{H^*},\theta\right)\right) + d\theta\sum_{f\in\mathcal{F}}\frac{\partial S}{\partial\theta}\left(\frac{T^f}{H^*},\theta\right) = 0.$$

Evaluating the above expression at $\theta = 0$, and using Lemma XVI and the equilibrium condition, we obtain:

$$-\frac{dH^*}{H^*(0)} - d\theta\sum_{f\in\mathcal{F}}\alpha S\left(\frac{T^f}{H^*(0)},0\right)^2 = 0,$$

which proves the first part of the lemma.

The second part of the lemma follows by Taylor's theorem:

$$\begin{aligned}\mathrm{CS}(\theta) - \mathrm{CS}(0) &= -\alpha\,\mathrm{HHI}(0)\theta + o(\theta),\\ &= -\alpha\,\mathrm{HHI}(\theta)\theta + o(\theta),\end{aligned}$$

where the second line follows from the fact that $\mathrm{HHI}(\theta) - \mathrm{HHI}(0)$ is at most first order. □

Let $\Pi(\theta)$ denote aggregate equilibrium profits when the conduct parameter is θ. We compute $\Pi'(0)$:

Lemma XVIII. $\Pi'(0) = \alpha^2\, HHI(0) \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(0)}, 0\right)$.

Proof. Let $\pi^f(\theta) = \alpha s^f(\theta)/(1-\alpha\theta s^f(\theta))$ denote firm f's equilibrium profit. Note that

$$\begin{aligned}{s^f}'(0) &= \left(-\frac{T^f}{H^*}\frac{d\log H^*}{d\theta}\frac{\partial S}{\partial(T^f/H^*)} + \frac{\partial S}{\partial\theta}\right)\Bigg|_{\theta=0},\\ &= \alpha\,\mathrm{HHI}(0)s^f(0) - \alpha(s^f(0))^2.\end{aligned}$$

Hence,

$${\pi^f}'(0) = \alpha\left({s^f}'(0) - s^f(0)\left(-\alpha s^f(0)\right)\right) = \alpha^2\,\mathrm{HHI}(0)s^f(0).$$

Adding up over all firms proves the lemma. □

Combining Lemmas XVII and XVIII, we obtain the second part of Proposition 4:

Lemma XIX. *In the neighborhood of $\theta = 0$,*

$$AS(\theta) - AS(0) = -\alpha\, HHI(\theta)\left(1 - \sum_{f\in\mathcal{F}} s^f(\theta)\right)\theta + o(\theta).$$

Proof. Lemmas XVII and XVIII and Taylor's theorem imply that

$$\mathrm{AS}(\theta) - \mathrm{AS}(0) = -\alpha\,\mathrm{HHI}(0)\left(1 - \sum_{f\in\mathcal{F}} s^f(0)\right)\theta + o(\theta).$$

The lemma follows from the fact that

$$\mathrm{HHI}(0)\left(1 - \sum_{f\in\mathcal{F}} s^f(0)\right) - \mathrm{HHI}(\theta)\left(1 - \sum_{f\in\mathcal{F}} s^f(\theta)\right)$$

is at most first order. □

IV.3 Proof of Proposition 6

Proof. Let $\mathrm{CS}(\theta)$ and $\mathrm{AS}(\theta)$ be pre-merger equilibrium consumer surplus and aggregate surplus, respectively. Let $\mathrm{HHI}(\theta)$ (resp., $H^*(\theta)$) be the pre-merger equilibrium value of the

Herfindahl index (resp., aggregator), and

$$\Sigma(\theta) \equiv \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*(\theta)}, \theta\right)$$

be the firms' aggregate market share. The post-merger values of those quantities are $\overline{\text{CS}}(\theta)$, $\overline{\text{AS}}(\theta)$, $\overline{\text{HHI}}(\theta)$, $\overline{H}^*(\theta)$, and $\overline{\Sigma}(\theta)$, respectively.

Note that $\text{CS}(0) = \overline{\text{CS}}(0)$, $\text{AS}(0) = \overline{\text{AS}}(0)$, $H^*(0) = \overline{H}^*(0)$, $\Sigma(0) = \overline{\Sigma}(0)$, and

$$\overline{\text{HHI}}(0) - \text{HHI}(0) = \Delta^M \text{HHI}(0),$$

where $\Delta^M \text{HHI}(\theta)$ is the merged-induced, naively-computed variation in the Herfindahl index.

Using these facts and Proposition 4, we obtain:

$$\begin{aligned}
\overline{\text{CS}}(\theta) - \text{CS}(\theta) &= -\alpha\left(\overline{\text{HHI}}(\theta) - \text{HHI}(\theta)\right)\theta + o(\theta),\\
&= -\alpha\left(\overline{\text{HHI}}(0) - \text{HHI}(0) + o(1)\right)\theta + o(\theta),\\
&= -\alpha\Delta^M \text{HHI}(0)\theta + o(\theta),\\
&= -\alpha\left(\Delta^M \text{HHI}(\theta) + o(1)\right)\theta + o(\theta),\\
&= -\alpha\Delta^M \text{HHI}(\theta)\theta + o(\theta),
\end{aligned}$$

which proves the first part of the proposition.

Similarly,

$$\begin{aligned}
\overline{\text{AS}}(\theta) - \text{AS}(\theta) &= -\alpha\Big(\overline{\text{HHI}}(\theta)\left(1 - \alpha\overline{\Sigma}(\theta)\right) - \text{HHI}(\theta)\left(1 - \alpha\Sigma(\theta)\right)\Big)\theta + o(\theta),\\
&= -\alpha\Big(\overline{\text{HHI}}(0)\left(1 - \alpha\Sigma(0)\right) - \text{HHI}(0)\left(1 - \alpha\Sigma(0)\right) + o(1)\Big)\theta + o(\theta),\\
&= -\alpha\left(1 - \alpha\Sigma(0)\right)\left(\overline{\text{HHI}}(0) - \text{HHI}(0)\right)\theta + o(\theta),\\
&= -\alpha\left(1 - \alpha\Sigma(\theta) + o(1)\right)\left(\Delta^M \text{HHI}(\theta) + o(1)\right)\theta + o(\theta),\\
&= -\alpha\left(1 - \alpha\Sigma(\theta)\right)\Delta^M \text{HHI}(\theta)\theta + o(\theta),
\end{aligned}$$

which proves the second part of the proposition. □

V Consumer Surplus Effects: Static Analysis

V.1 Proof of Proposition 8

Proof. Recall that $\varepsilon(\cdot)$ is the elasticity of S (see Lemma I) and that the cutoff type solves the equation:

$$S\left(\frac{\hat{T}^M}{H^*}\right) = \sum_{f\in\mathcal{M}} S\left(\frac{T^f}{H^*}\right).$$

Totally differentiating this equation, we obtain:

$$\begin{aligned}
S'\left(\frac{\hat{T}^M}{H^*}\right)\frac{d\hat{T}^M}{dH^*} &= \frac{\hat{T}^M}{H^*}S'\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f\in\mathcal{M}}\frac{T^f}{H^*}S'\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)S\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f\in\mathcal{M}}\varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right)\sum_{f\in\mathcal{M}}S\left(\frac{T^f}{H^*}\right) - \sum_{f\in\mathcal{M}}\varepsilon\left(\frac{T^f}{H^*}\right)S\left(\frac{T^f}{H^*}\right), \\
&= \sum_{f\in\mathcal{M}}\left(\varepsilon\left(\frac{\hat{T}^M}{H^*}\right) - \varepsilon\left(\frac{T^f}{H^*}\right)\right)S\left(\frac{T^f}{H^*}\right), \\
&< 0,
\end{aligned}$$

where the third line follows by definition of $\hat{T}^M$ and the last line follows from Lemma I and from the fact that $\hat{T}^M > T^f$ for every $f \in \mathcal{M}$. □

V.2 Proof of Proposition 9

Proof. Note that

$$\frac{\hat{T}^M}{T^f + T^g} = \frac{S^{-1}\left(S\left(\frac{T^f}{H^*}\right) + S\left(\frac{T^g}{H^*}\right)\right)}{\frac{T^f}{H^*} + \frac{T^g}{H^*}} = \xi\left(\frac{T^f}{H^*}, \frac{T^g}{H^*}\right),$$

where

$$\xi(x, y) \equiv \frac{S^{-1}\left(S(x) + S(y)\right)}{x + y}, \quad \forall x, y > 0.$$

Proving the proposition therefore boils down to showing that $\partial\xi/\partial x > 0$ and $\partial\xi/\partial y > 0$. By symmetry, this is equivalent to proving that $\partial\xi/\partial x > 0$, which we undertake next.

Differentiating ξ with respect to x, we obtain:

$$\frac{\partial\xi}{\partial x} = \frac{S^{-1}(S(x) + S(y))}{(x + y)^2}\left(\underbrace{\frac{(x + y) \times S'(x)}{S^{-1}(S(x) + S(y)) \times S' \circ S^{-1}(S(x) + S(y))}}_{\equiv\psi(x,y)} - 1\right).$$

Let $z = S^{-1}(S(x) + S(y))$. By definition, $S(z) = S(x) + S(y)$. Moreover, by subadditivity of S, $z > x + y$. Assume first that $x \le y$. Note that

$$\psi(x, y) = \frac{(x + y)S'(x)}{zS'(z)},$$

$$
\begin{aligned}
&= \frac{(x+y)S'(x)/(S(x)+S(y))}{zS'(z)/S(z)}, \\
&= \frac{\frac{xS'(x)}{S(x)}\frac{S(x)}{S(x)+S(y)} + \frac{yS'(x)}{S(y)}\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \\
&\geq \frac{\frac{xS'(x)}{S(x)}\frac{S(x)}{S(x)+S(y)} + \frac{yS'(y)}{S(y)}\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \text{ by concavity of } S \text{ (see Lemma I)}, \\
&= \frac{\varepsilon(x)\frac{S(x)}{S(x)+S(y)} + \varepsilon(y)\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \\
&> \frac{\varepsilon(z)\frac{S(x)}{S(x)+S(y)} + \varepsilon(z)\frac{S(y)}{S(x)+S(y)}}{\varepsilon(z)}, \text{ since } \varepsilon \text{ is decreasing (see Lemma I)}, \\
&= 1.
\end{aligned}
$$

Therefore, $\partial\xi/\partial x > 0$ whenever $x \leq y$.

Next, assume for a contradiction that $\psi(x,y) \leq 1$ for some $x > y$. Take the smallest such x. By continuity, this x exists, and satisfies $x > y$ (as shown in the first step of the proof) and $\psi(x,y) = 1$. Note that

$$
\begin{aligned}
\frac{\partial\psi}{\partial x} &= \frac{1}{(zS'(z))^2}\left((S'(x)+(x+y)S''(x))\,zS'(z) - (x+y)S'(x)\left(S'(x)+S'(x)\frac{zS''(z)}{S'(z)}\right)\right), \\
&= \frac{1}{(zS'(z))^2}\left((x+y)S''(x)zS'(z) - (x+y)(S'(x))^2\frac{zS''(z)}{S'(z)}\right), \text{ since } \psi(x,y)=1, \\
&= \frac{(x+y)z}{(zS'(z))^2}\left(S''(x)S'(z) - (S'(x))^2\frac{S''(z)}{S'(z)}\right), \\
&= \frac{(x+y)z(S'(x))^2S'(z)}{(zS'(z))^2}\left(\frac{S''(x)}{(S'(x))^2} - \frac{S''(z)}{(S'(z))^2}\right).
\end{aligned}
$$

Next, we argue that $S''(\cdot)/(S'(\cdot))^2$ is decreasing. Recall from Lemma I that

$$
S'(x) = \frac{1}{x}\frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}.
$$

It follows that

$$
S''(x) = -\frac{\alpha(2-S(x))(1-S(x))(1-\alpha S(x))S(x)^2}{x^2\left(1-S(x)1+\alpha S(x)^2\right)^3}.
$$

Hence,

$$
\frac{S''(x)}{(S'(x))^2} = -\frac{\alpha(2-S(x))}{(1-S(x))(1-\alpha S(x))\left(1-S(x)1+\alpha S(x)^2\right)}.
$$

Since $S(\cdot)$ is strictly increasing, the above expression is strictly decreasing in x if and only if

$$
\varphi(s) = \frac{\alpha(2-s)}{(1-s)(1-\alpha s)\left(1-s1+\alpha s^2\right)}
$$

is strictly increasing in s. Routine calculations show that $\varphi'(s) > 0$ for every $s \in (0,1)$ and $\alpha \in (0,1]$. Therefore, $\partial\psi(x,y)/\partial x > 0$. It follows that $\psi(x',y) < 1$ in a small neighborhood to the left of x. This contradicts the definition of x. We can conclude that ξ is increasing in both of its arguments, which proves the proposition.[2] $\square$

VI Consumer Surplus Effects: Dynamic Analysis

VI.1 Proof of Corollary 1

Proof. The corollary is the analogue of Lemma 4 in Nocke and Whinston (2010), and its proof is identical to that of the lemma in the earlier paper. It suffices to make the following two observations.

First, Lemma 4 in Nocke and Whinston (2010) states the result for the "most lenient" myopically CS-maximizing merger policy. However, the result and proof also hold for the "least lenient" such policy. As noted in the text, these two policies are generically identical in our model as every merger is, generically, either CS-increasing or CS-decreasing, but not CS-neutral.

Second, the proof of Lemma 4 uses the monotonicity property of Lemma 2 in Nocke and Whinston (2010). It is straightforward to see that Lemmas 5 and 6 in Nocke and Whinston (2010) hold in our setup, implying that the monotonicity property of Lemma 2 carries over as well. $\square$

VI.2 Proof of Proposition 12

Proof. We first show that merger M_k is profitable if it is CS-neutral. Recall that the profit of a firm can be written as $\Pi = m - 1$, and its market share as $S = (m-1)/(\alpha m)$. It follows that $\Pi = \alpha m S$. Note that

$$m\left(\frac{T^{M_k}}{H^*}\right) S\left(\frac{T^{M_k}}{H^*}\right) = m\left(\frac{T^{M_k}}{H^*}\right) \sum_{f\in\mathcal{M}_k} S\left(\frac{T^f}{H^*}\right) > \sum_{f\in\mathcal{M}_k} m\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right),$$

where the equality follows because the merger is CS-neutral, and the inequality follows because $\hat{T}^{M_k} > T^f$ for every $f \in \mathcal{M}_k$ and $m'(\cdot) > 0$.

Hence, merger M_k is profitable if $T^{M_k} = \hat{T}^{M_k}$. Next, suppose that the merger is CS-increasing, i.e., $T^M > \hat{T}^M$. Then, by Proposition 2, the merged firm makes a strictly higher equilibrium profit than if its type were $\hat{T}^{M_k}$, i.e., if it were CS-neutral. $\square$

[2] To see why $\hat{T}^M - (T^f + T^g) > \hat{T}^{M'} - (T^{f'} + T^{g'})$ (as mentioned in footnote 26 in the paper), note that

$$\frac{\hat{T}^M - (T^f + T^g)}{T^{f'} + T^{g'}} > \frac{\hat{T}^M - (T^f + T^g)}{T^f + T^g} > \frac{\hat{T}^{M'} - (T^{f'} + T^{g'})}{T^{f'} + T^{g'}},$$

where the first inequality follows from the fact that $T^f + T^g > T^{f'} + T^{g'}$ and the second inequality follows from the first part of the proposition.

VI.3 Proof of Proposition 14

Proof. The proposition is the analogue of Proposition 3, part (i) in Nocke and Whinston (2010), and its proof is identical to that of the proposition in the earlier paper. (Note that, in our model, the most and least lenient myopically CS-maximizing merger policies generically coincide.) The proof in Nocke and Whinston (2010) makes explicit use of the statement about the private profitability of CS-nondecreasing mergers in Corollary 1 as well as of Lemmas 2, 4 and 5 in that paper. The profitability statement of Corollary 1 in Nocke and Whinston (2010) corresponds to Proposition 12 in our paper whereas Lemma 4 in Nocke and Whinston (2010) corresponds to our Corollary 1. As noted in the proof of our Corollary 1, Lemmas 5 and 6 in Nocke and Whinston (2010) hold in our setup, implying that Lemma 2 in Nocke and Whinston (2010) carries over as well. □

VII External Effects

VII.1 Preliminaries

We first derive the formula for $\eta(H)$:

Lemma XX. *$\eta(H)$ is given by:*

$$\eta(H) = -1 + \sum_{f \in \mathcal{O}} \phi(s^f, \alpha),$$

where $s^f = S(T^f/H)$, and

$$\phi(s, \alpha) = \frac{\alpha s(1-s)}{(1-\alpha s)(1 - s + \alpha s^2)}, \quad \forall s \in (0,1), \ \forall \alpha \in (0,1].$$

Proof. This follows from the definition of η and from the fact that

$$\begin{aligned} xm'(x) &= x\alpha \frac{S'(x)}{(1-\alpha S(x))^2}, \text{ since } m(x) = \frac{1}{1-\alpha S(x)}, \\ &= \frac{\alpha}{(1-\alpha S(x))^2} \frac{S(x)(1-S(x))(1-\alpha S(x))}{1 - S(x) + \alpha S(x)^2}, \text{ by Lemma I}, \\ &= \frac{\alpha S(x)(1-S(x))}{(1-\alpha S(x))(1-S(x)+\alpha S(x)^2)}, \\ &= \phi(S(x), \alpha). \end{aligned}$$

□

Next, we put on record the following facts about the function ϕ:

Lemma XXI. *Let $\hat{\alpha} = \frac{1}{2} + \frac{\sqrt{33}}{18} \simeq 0.82$. The function ϕ has the following properties:*

(a) For every $s \in (0,1)$, $\phi(s, \cdot)$ is strictly increasing.

(b) If $\alpha \leq \hat{\alpha}$, then $\phi(s, \alpha) \leq s$ for every $s \in (0, 1)$.

Moreover, if $\alpha > \hat{\alpha}$, then there exist thresholds $s^*(\alpha) \in (0, 1]$ and $\hat{s}(\alpha) \in (1/4, 1)$ such that:

(c) $\phi(\cdot, \alpha)$ is strictly increasing on $(0, s^*(\alpha))$ and strictly decreasing on $(s^*(\alpha), 1)$.

(d) $\phi(\cdot, \alpha)$ is strictly convex on $(0, \hat{s}(\alpha))$ and strictly concave on $(\hat{s}(\alpha), 1)$.

Proof. We prove the lemma (analytically) using Mathematica. Mathematica files are available upon request. □

VII.2 Proof of Proposition 16

Proof. If $\alpha \leq \hat{\alpha}$, then, by Lemma XXI, $\phi(x, \alpha) \leq x$ for every $x \in (0, 1)$. As outsiders' market shares add up to strictly less than 1, Lemma XX immediately implies that any infinitesimal CS-decreasing merger has a negative external effect. Hence, any (not necessarily infinitesimal) CS-decreasing merger has a negative external effect.

Next, suppose $\alpha > \hat{\alpha}$, and define

$$\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}^n, \text{ where } \mathcal{S}^n = \{s \in [0, 1]^n : \sum_{i=1}^n s_i \leq 1\} \ \forall n \geq 1,$$

$$\bar{\mathcal{S}} = \bigcup_{n \geq 1} \mathcal{S}^n, \text{ where } \bar{\mathcal{S}}^n = \{s \in [0, 1]^n : \sum_{i=1}^n s_i = 1\} \ \forall n \geq 1,$$

and

$$\Psi(\alpha) = \sup_{s \in \mathcal{S}} \sum_s \phi(\cdot, \alpha), \ \forall \alpha \in (\hat{\alpha}, 1],$$

where

$$\sum_s \phi(\cdot, \alpha) \equiv \sum_{i=1}^n \phi(s_i, \alpha), \ \forall s = (s_i)_{1 \leq i \leq n} \in \mathcal{S}, \ \forall \alpha \in (0, 1].$$

Clearly, since $\phi(x, \alpha) \geq 0$ for all x, we have that $\Psi(\alpha) = \sup_{s \in \bar{\mathcal{S}}} \sum_s \phi(\cdot, \alpha)$. Next, we claim that $\Psi(\alpha) = \sup_{s \in \bar{\mathcal{S}}^4} \sum_s \phi(\cdot, \alpha)$. To prove this, we show that, for every $s \in \bar{\mathcal{S}}$, there exists $s' \in \bar{\mathcal{S}}^4$ such that

$$\sum_s \phi(\cdot, \alpha) \leq \sum_{s'} \phi(\cdot, \alpha).$$

If s belongs to $\mathcal{S}^n$ for some $n \leq 4$, or, more generally, if s has at most four components different from zero, then this is obvious. Assume instead that s has five or more components different from zero. Assume without loss of generality that $s \in \bar{\mathcal{S}}^n$ for some $n \geq 5$, that $s_i > 0$ for every i, and that the components of s_i have been sorted in increasing order. We construct s' by induction.

Let us first define a function ξ, which takes as argument a profile of market shares $\tilde{s} \in \bar{\mathcal{S}}^n$ sorted in increasing order and with strictly positive components, and returns a profile of market shares $\xi(\tilde{s})$ sorted in increasing order and with strictly positive components, such that either $\xi(\tilde{s}) \in \bar{\mathcal{S}}^n$, or $\xi(\tilde{s}) \in \bar{\mathcal{S}}^{n-1}$. $\xi(\tilde{s})$ is defined as follows:

- If $\tilde{s}_2 \geq \hat{s}(\alpha)$ (or if $\tilde{s} \in \mathcal{S}^1$), then $\xi(\tilde{s}) = \tilde{s}$.
- If $\tilde{s}_2 < \hat{s}(\alpha)$, then do the following:
  - If $\tilde{s}_1 + \tilde{s}_2 \leq \hat{s}(\alpha)$, then form the $(n-1)$-dimensional vector with first component $\tilde{s}_1 + \tilde{s}_2$ and remaining components $(\tilde{s}_i)_{3 \leq i \leq n}$, and sort that vector in increasing order to obtain $\xi(\tilde{s})$.
  - If instead $\tilde{s}_1 + \tilde{s}_2 > \hat{s}(\alpha)$, then form the n-dimensional with first component $\tilde{s}_1 + \tilde{s}_2 - \hat{s}(\alpha)$, second component $\hat{s}(\alpha)$, and remaining components $(\tilde{s}_i)_{3 \leq i \leq n}$, and sort that vector in increasing order to obtain $\xi(\tilde{s})$.

Note that, since $\phi_\alpha(\cdot)$ is convex on $[0, \hat{s}(\alpha)]$, we have that, for every $\tilde{s} \in \bar{\mathcal{S}}$

$$\sum_{\tilde{s}} \phi(\cdot, \alpha) \leq \sum_{\xi(\tilde{s})} \phi(\cdot, \alpha).$$

We can now define the sequence $(s^k)_{k \geq 0}$ by induction: $s^0 = s$; $s^{k+1} = \xi(s^k)$ for every $k \geq 0$. Let m^k denote the number of components of s^k greater or equal to $\hat{s}(\alpha)$, and n^k denote the dimensionality of the vector s^k. By definition of ξ and of the sequence $(s^k)_{k \geq 0}$, the sequence of integers $(m^k)_{k \geq 0}$ (resp. $(n^k)_{k \geq 0}$) is non-decreasing (resp. non-increasing) and bounded above by n (resp. bounded below by 1). Therefore, those sequences of integers are eventually stationary: There exists $K \geq 0$ such that $m^k = m^{k+1}$ and $n^k = n^{k+1}$ for every $k \geq K$. It follows that $(s^k)_{k \geq 0}$ is also stationary after K. Let s' be the stationary value of the sequence $(s^k)_{k \geq 0}$. Then, by induction on k,

$$\sum_{s} \phi(\cdot, \alpha) \leq \sum_{s'} \phi_\alpha(\cdot, \alpha).$$

Moreover, s' has at most one component in $[0, \hat{s}(\alpha))$ (for otherwise, $\xi(s')$ would not be equal to s'). Let n' be the dimensionality of the vector s'. We claim that $n' \leq 4$. Suppose $n' > 1$. Then,

$$1 = \sum_{i=1}^{n'} s'_i \geq (n'-1)\hat{s}(\alpha) > \frac{1}{4} \times (n'-1),$$

where the last inequality follows by Lemma XXI. Hence, $n' \leq 4$. Having constructed s', we can conclude that

$$\Psi(\alpha) = \sup_{s \in \bar{\mathcal{S}}^4} \sum_{s} \phi_\alpha(\cdot, \alpha). \tag{xiv}$$

By continuity of $\phi(\cdot, \alpha)$ (or, rather, of $\phi(\cdot, \alpha)$'s continuous extension to $[0, 1]$) and compactness of $\bar{\mathcal{S}}^4$, the maximization problem defined in equation (xiv) has a solution. Let s be such a solution. Then, by the convexity argument used in the construction of s', s has a most one component in $(0, \hat{s}(\alpha))$. Moreover, since $\phi(\cdot, \alpha)$ is strictly concave on $[\hat{s}(\alpha), 1]$, the components of s that are greater or equal to $\hat{s}(\alpha)$ must be equal to each other. It follows that

$$\Psi(\alpha) = \max_{x \in [0,1]} \max \left(\phi(x, \alpha) + \phi(1-x, \alpha), \phi(x, \alpha) + 2\phi\left(\frac{1-x}{2}, \alpha\right), \phi(x, \alpha) + 3\phi\left(\frac{1-x}{3}, \alpha\right) \right).$$

We (analytically) solve the above maximization problem using Mathematica. We obtain:

$$\Psi(\alpha) = \begin{cases} \frac{18\alpha}{18-3\alpha-\alpha^2} & \text{if } \alpha \leq \frac{6}{7}, \\ \frac{4\alpha}{4-\alpha^2} & \text{otherwise.} \end{cases}$$

It is straightforward to check that Ψ is strictly increasing, and that $\Psi(\hat{\alpha}) < 1 < \Psi(1)$. The unique solution of equation $\Psi(\alpha) = 1$ on the interval $(\hat{\alpha}, 1]$ is $\bar{\alpha} = \frac{3}{2}(\sqrt{57} - 7)$.

We can conclude. Assume first that $\alpha \in (\hat{\alpha}, \bar{\alpha}]$. Then, for every profile of outsiders' market shares $(s^f)_{f \in \mathcal{O}}$,

$$\sum_{f \in \mathcal{O}} \phi(s^f, \alpha) < \phi\left(1 - \sum_{f \in \mathcal{O}} s^f, \alpha\right) + \sum_{f \in \mathcal{O}} \phi(s^f, \alpha) \leq \Psi(\alpha) \leq \Psi(\bar{\alpha}) = 1.$$

Therefore, any CS-decreasing merger must have a negative external effect.

Assume instead that $\alpha > \bar{\alpha}$. We first show that there exists an infinitesimal CS-decreasing merger that has a negative external effect. Let $\mathcal{O} = \{1\}$ and $\mathcal{I} = \{2, 3\}$. Since $\phi(\cdot, \alpha)$ is continuous and $\phi(0, \alpha) = 0$, there exists $s \in (0, 1)$ such that $\phi(s, \alpha) < 1$. Let $T^1 = S^{-1}(s)$, $T^2 = T^3 = S^{-1}((1-s)/2)$, and $H^0 = 0$. Then, by construction, the pre-merger equilibrium aggregator level is $H = 1$, and market shares are as follows: $s^1 = s$, $s^2 = s^3 = (1-s)/2$. The external effect of an infinitesimal CS-decreasing merger between firms 2 and 3 is given by $\phi(s, \alpha) - 1$, which is strictly negative by construction.

Next, we claim that there exists an infinitesimal CS-decreasing merger that has a positive external effect. Since $\Psi(\alpha) > 1$, there exists $(s_i)_{1 \leq i \leq n} \in (0, 1]^n$ such that $\sum_{i=1}^n s_i \leq 1$ and $\sum_{i=1}^n \phi(s_i, \alpha) > 1$. By continuity, for $\varepsilon > 0$ small enough, $\sum_{i=1}^n \phi(s_i - \varepsilon, \alpha) > 1$. Let $\mathcal{O} = \{1, \ldots, n\}$, $\mathcal{I} = \{n+1, n+2\}$, $s'^i = s_i - \varepsilon$ for every $i \in \mathcal{O}$, $s'^i = \frac{1}{2}\left(1 - \sum_{j=1}^n s'^j\right)$ for $i \in \mathcal{I}$, $T^i = S^{-1}(s'^i)$ for every $i \in \mathcal{I} \cup \mathcal{O}$, and $H^0 = 0$. Then, by construction, an infinitesimal CS-decreasing merger between the insiders has a positive external effect.

Since any CS-decreasing merger can be decomposed into the integral of infinitesimal CS-decreasing mergers, and since a CS-decreasing merger can be made infinitesimal by tweaking the post-merger type of the merged entity, the above existence results extend immediately to non-infinitesimal mergers: If $\alpha > \bar{\alpha}$, then there exist CS-decreasing mergers that have a positive external effect, and CS-decreasing mergers that have a negative external effect. □

VII.3 Proof of Proposition 17

Proof. **(i).** It is easy to show that $s^* \equiv \inf_{\alpha\in[\bar{\alpha},1]} s^*(\alpha) \simeq 0.68$, where $s^*(\alpha)$ was defined in Lemma XXI. Let $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$ such that $s \geq_1 s'$, and $s^f \leq s^*$ for every $f \in \mathcal{O}$. There exists an injection $\iota : \mathcal{O}' \longrightarrow \mathcal{O}$ such that $s^{\iota(f)} \geq s'^f$ for every $f \in \mathcal{O}'$. Note that

$$-1 + \sum_{f\in\mathcal{O}'} \phi(s'^f) \leq -1 + \sum_{f\in\mathcal{O}'} \phi(s^{\iota(f)}) \leq -1 + \sum_{f\in\mathcal{F}} \phi(s^f, \alpha),$$

where the first inequality follows by Lemma XXI, and the second inequality follows by injectivity of ι and non-negativity of ϕ. This proves the first part of the proposition.

(ii) It is easy to show that $\hat{s} \equiv \inf_{\alpha\in[\bar{\alpha},1]} \hat{s}(\alpha) \simeq 0.29$, where $\hat{s}(\alpha)$ was defined in Lemma XXI. Let $s = (s^f)_{f\in\mathcal{O}}$ and $s' = (s'^f)_{f\in\mathcal{O}'}$ such that $s \geq_2 s'$, $s^f \leq \hat{s}$ for every $f \in \mathcal{O}$, and $s'^f \leq \hat{s}$ for every $f \in \mathcal{O}'$. Since $s \geq_2 s'$, those vectors have the same length, and we can assume that $\mathcal{O} = \mathcal{O}' = \{1, \dots, n\}$ without loss of generality. Note that

$$\begin{aligned} -1 + \sum_{f=1}^{n} \phi(s^f, \alpha) &= -1 + n \int_0^{\hat{s}} \phi(x, \alpha) dP_s(x), \\ &\geq -1 + n \int_0^{\hat{s}} \phi(x, \alpha) dP_{s'}(x), \\ &= -1 + \sum_{f=1}^{n} \phi(s'^f, \alpha), \end{aligned}$$

where the inequality follows from the convexity of $\phi(\cdot, \alpha)$ on $[0, \hat{s}]$ (see Lemma XXI), and the fact that $\int_0^{\hat{s}} x dP_s(x) = \int_0^{\hat{s}} x dP_{s'}(x)$ and $P_{s'}$ second-order stochastically dominates P_s. This proves the second part of the proposition. □

VIII Competition Within and Across Nests

Throughout the paper, we have confined attention to the case where the nest partition $\mathcal{L}$ is a filtration of the firm partition $\mathcal{F}$. As was made clear in the main text, this restriction delivers a multiproduct-firm pricing game that is aggregative, which permits a tractable merger analysis. The price to pay for this tractability is that, by design, competition takes place across nests only. Moreover, the products offered by any two firms f' and f'' are equally close substitutes to those offered by any third firm f. In the following, we provide a merger analysis in a framework that allows competition to take place both within and between nests. The framework also allows some firms to offer closer or more distant substitutes to some other firms' products.

We extend the oligopoly model of Section 2 as follows. The demand system continues to be of the NCES or NMNL type. We partition the set of nests $\mathcal{L}$ into two subsets: $\mathcal{L}^b$ and $\mathcal{L}^n$.

The set $\mathcal{L}^b$ is further partitioned into a set of "broad" firms $\mathcal{F}^b$. As in the model of Section 2, each broad firm $f \in \mathcal{F}^b$ owns exclusive property rights over all the products in one or several nests.

The novelty compared to Section 2 is that each nest $l \in \mathcal{L}^n$ is partitioned into a set of "narrow" firms $\mathcal{F}_l$. By assumption, the products of a given narrow firm $f \in \mathcal{F}_l$ are all contained in nest l. We assume that the partition $\mathcal{F}_l$ contains at least two elements for every $l \in \mathcal{L}^n$. (If $\mathcal{F}_l$ were a singleton, we would classify the corresponding firm as a broad firm.) In the following, when studying a narrow firm $f \in \mathcal{F}_l$, we will often write $f \in l$ with a slight abuse of notation. The set of narrow firms is denoted $\mathcal{F}^n \equiv \bigcup_{l \in \mathcal{L}^n} \mathcal{F}_l$.

The set of firms is $\mathcal{F} \equiv \mathcal{F}^b \cap \mathcal{F}^n$. The difference between narrow and broad firms is that narrow firms face competition from rivals both within and outside their nests, whereas broad firms only face competition from rivals outside their nests. Note that in the special case where $\mathcal{F}^n = \emptyset$, the oligopoly model reduces to the model of Section 2.

As will soon become clear, the restriction that each firm is either narrow or broad ensures that the oligopoly game retains some aggregative properties in the following sense: The behavior of broad firm f depends only on the value of the industry aggregator H, whereas the behavior of narrow firm $f \in l$ depends solely on the values of the industry aggregator H and the nest-level aggregator H_l. When studying mergers, we need to ensure that the post-merger oligopoly model continues to satisfy the restriction that each firm is either narrow or broad. This means that we need to confine attention to the following types of mergers: Mergers between broad firms such that the merged entity is also a broad firm; mergers between narrow firms operating in the same nest l such that all of the merged firms' products are in nest l.

The remainder of this section is organized as follows. In Section VIII.1, we characterize the unique equilibrium of the oligopoly model as the solution to a nested fixed-point problem, show that the type aggregation property continues to hold, and perform comparative statics. In Section VIII.2, we make use of the type aggregation property to provide a simple conceptual framework to model mergers. Section VIII.3 develops a static analysis of the consumer surplus effects of mergers and shows that all the results derived in Section 4.1 continue to hold. Section VIII.4 studies the consumer surplus effects of mergers in a dynamic framework where merger opportunities arise stochastically over time, and shows that all the results derived in Section 4.2 continue to hold. Section VIII.5 discusses whether mergers between narrow firms raise more competitive concerns than mergers between broad firms. Short proofs are provided in the main text. Longer mathematical developments are relegated to Section VIII.6.

VIII.1 The Oligopoly Model: Equilibrium Analysis

We know from Appendix I that for broad firms, first-order conditions are sufficient for global optimality. This implies that the behavior of broad firm $f \in \mathcal{F}^b$ with type T^f can still be described by the markup, market-share, and profit fitting-in functions $m(T^f/H)$, $S(T^f/H)$ and $\pi(T^f/H)$ defined in Section 2.3.

Moreover, we know from Lemma XXI in the Online Appendix to Nocke and Schutz (2018) that first-order conditions are also sufficient for optimality for narrow firms. In the following, we use this to define firm- and nest-level fitting-in functions, establish equilibrium uniqueness, and characterize the equilibrium as a nested fixed-point problem. We then perform comparative statics.

Firm-level fitting-in functions. Let f be a narrow firm operating in nest l. We know from Lemma XXI in the Online Appendix to Nocke and Schutz (2018) that the optimal prices of firm f satisfy the common ι-markup property. Let $\widetilde{\mu}^f$ be the ι-markup of firm $f \in l$. Equation (xxix) in the Online Appendix to Nocke and Schutz (2018), which characterizes firm f's optimal ι-markup as a function of H_l and H, can be rewritten as follows:[3]

$$\frac{\widetilde{\mu}^f - 1}{\widetilde{\mu}^f} = (1-\beta)\widetilde{\alpha}\frac{\sum_{j\in f} h_j}{H_l} + \beta\widetilde{\alpha}\frac{\sum_{j\in f} h_j}{H_l}\frac{H_l^\beta}{H}, \tag{xv}$$

where, as in Section 2.2, $\widetilde{\alpha}$ is equal to $(\sigma-1)/\sigma$ under NCES demand and to 1 under NMNL demand.

Define firm f's market share within nest l as

$$\widetilde{s}^f = \frac{\sum_{j\in f} h_j}{H_l}.$$

In the discrete/continuous choice micro-foundation, $\widetilde{s}^f$ is the probability that the consumer chooses a product sold by firm f conditional on having chosen nest l. The market share of nest l, which also corresponds to the probability that nest l is chosen, is given by

$$s_l = \frac{H_l^\beta}{H}.$$

Firm f's market share at the industry level is therefore given by $s^f \equiv \widetilde{s}^f s_l$. As Section 2, market shares are measured in value in the NCES case and in volume in the NMNL case.

Having defined market shares, we can rewrite equation (xv) as follows:

$$\widetilde{\mu}^f = \frac{1}{1 - \widetilde{\alpha}\widetilde{s}^f\left(1-\beta+\beta s_l\right)}. \tag{xvi}$$

Intuitively, firm f sets a high markup if it has a high market share in its nest or if its nest commands a high market share at the industry level.

Moreover, it is straightforward to show that

$$\sum_{j\in f} h_j = \left(T^f\right)^{\frac{1}{\beta}} \psi(\widetilde{\mu}^f),$$

[3] To see how to derive equation (xv) from equation (xxix) in Nocke and Schutz (2018), note that under NCES or NMNL demand, $\Phi_l(H_l) = H_l^\beta$, $\Psi(H) = \log(H^0 + H)$, and $\gamma_j = \alpha h_j$.

where

$$\psi(\widetilde{\mu}^f) = \begin{cases} \left(1-(1-\widetilde{\alpha})\widetilde{\mu}^f\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}} & \text{under NCES demand,} \\ \mathrm{e}^{-\widetilde{\mu}^f} & \text{under NMNL demand,} \end{cases}$$

and

$$T^f = \begin{cases} \left(\sum_{j\in f} a_j c_j^{1-\sigma}\right)^{\beta} & \text{under NCES demand,} \\ \left(\sum_{j\in f} \exp \frac{a_j - c_j}{\lambda}\right)^{\beta} & \text{under NMNL demand.} \end{cases}$$

Firm f's type, T^f, has the same interpretation as in the main text: Consumer surplus would be equal to $\log T^f$ if firm f were to price all of its products at marginal cost and no other firm were present in the industry.

Firm f's market share in nest l can be rewritten as follows:

$$\widetilde{s}^f = \frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l}\psi(\widetilde{\mu}^f). \tag{xvii}$$

It is straightforward to show that the system of equations (xvi)–(xvii) has a unique solution in $(\widetilde{\mu}^f, \widetilde{s}^f) \in (1, 1/(1-\widetilde{\alpha})) \times \mathbb{R}_{++}$, which we denote[4]

$$\left(\widetilde{m}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l}, s_l\right), \widetilde{S}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l}, s_l\right)\right).$$

Clearly, $\widetilde{m}$ and $\widetilde{S}$ are smooth on $\mathbb{R}^2_{++}$. Moreover, $\widetilde{m}(\cdot,\cdot)$ is strictly increasing in both arguments, and $\widetilde{S}(\cdot)$ is strictly increasing in the first argument and strictly decreasing in the second one. For some of the proofs, we will also require information on the range of $\widetilde{S}(\cdot, s_l)$ for every $s_l > 0$: We have that $\lim_{x\downarrow 0} \widetilde{S}(x, s_l) = 0$ and $\lim_{x\to\infty} \widetilde{S}(x, s_l) \geq 1$. Finally, since $1 < \widetilde{m}(\cdot,\cdot) < 1/(1-\widetilde{\alpha})$, we have that $(1-\beta+\beta s_l)\widetilde{S}(\cdot, s_l) < 1$ for every $s_l > 0$.

Nest-level fitting-in functions. Let $l \in \mathcal{L}^n$. The monotonicity properties of $\widetilde{S}$ and the fact that $\lim_{x\downarrow 0} \widetilde{S}(x, s_l) = 0$ and $\lim_{x\to\infty} \widetilde{S}(x, s_l) \geq 1$ imply that, for given $H > 0$, there is a unique $H_l > 0$ such that

$$\sum_{f\in l} \widetilde{S}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^{\beta}}{H}\right) = 1. \tag{xviii}$$

(See Lemma XXIII in the Online Appendix to Nocke and Schutz (2018) for a more general version of this result.) Let $H_l(H)$ be the unique solution to this equation. The monotonicity properties of $\widetilde{S}$ also imply that $H_l(\cdot)$ is strictly increasing. The function $H_l(\cdot)$ allows us to

[4]In the NMNL case ($\alpha = 1$), the upper bound $1/(1-\alpha)$ is equal to ∞.

define the nest-market-share fitting-in function $\Sigma_l(\cdot)$ as

$$\Sigma_l(H) \equiv \frac{\left(H_l(H)\right)^\beta}{H}. \tag{xix}$$

The argument in the proof of Lemma XXIV in the Online Appendix to Nocke and Schutz (2018) implies that $\Sigma_l(\cdot)$ is strictly decreasing, $\Sigma_l(H) > 1$ for H sufficiently close to 0, and $\lim_{H\to\infty} \Sigma_l(H) = 0$.

Equilibrium condition. The analogue of equilibrium condition (12) in the main text is:

$$\frac{H^0}{H} + \sum_{f\in\mathcal{F}^b} S\left(\frac{T^f}{H}\right) + \sum_{l\in\mathcal{L}^n} \Sigma^l(H) = 1, \tag{xx}$$

i.e., the market share of the outside option, the market shares of broad firms, and the market shares of the nests of narrow firms add up to unity. The properties of the functions $\widetilde{S}(\cdot)$ (derived in Section 2.3) and $\Sigma_l(\cdot)$ (derived above) imply that equation (xx) has a unique solution, which pins down the equilibrium aggregator level H^*. Hence, there is a unique equilibrium.

Profits. It is straightforward to show that the profit of narrow firm $f \in l$, π^f, is given by

$$\pi^f = \widetilde{\alpha}\beta\widetilde{\mu}^f\widetilde{s}^f s^l.$$

Thus, firm f's profit fitting-in function is given by:

$$\pi^f(H) = \widetilde{\alpha}\beta\widetilde{m}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l(H)}, \Sigma_l(H)\right)\widetilde{S}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l(H)}, \Sigma_l(H)\right)\Sigma_l(H). \tag{xxi}$$

The formula in the statement of Theorem III in the Online Appendix to Nocke and Schutz (2018) also implies that

$$\pi^f(H) = \left(\widetilde{m}\left(\frac{\left(T^f\right)^{\frac{1}{\beta}}}{H_l(H)}, \Sigma_l(H)\right) - 1\right)\frac{1}{1 + \frac{1-\beta}{\beta}\frac{1}{\Sigma_l(H)}}. \tag{xxii}$$

We summarize these insights in the following proposition:

Proposition I. *A multiproduct-firm pricing game with broad and narrow firms has a unique equilibrium. The equilibrium aggregator level H^* is the unique solution of equation* (xx). *The behavior of broad firm $f \in \mathcal{F}^b$ is governed by the fitting-in functions $m(\cdot)$, $S(\cdot)$, and $\pi(\cdot)$. The behavior of narrow firm $f \in l$ is governed by the fitting-in functions $\widetilde{m}(\cdot,\cdot)$, $\widetilde{S}(\cdot,\cdot)$, $\pi^f(\cdot)$, $H_l(\cdot)$, and $\Sigma_l(\cdot)$.*

Comparative statics. The following comparative statics, which may be of independent interest, will play an important role in our merger analysis:

Proposition II. *Consider a multiproduct-firm pricing game with broad and narrow firms. Let $l \in \mathcal{L}^n$ and $f \in \mathcal{F}_l$. In equilibrium, an increase in T^f*

(i) raises H^*,

(ii) raises H_l and s_l,

(iii) raises $H_{l'}$ and lowers $s_{l'}$ for every $l' \in \mathcal{L}^n$ such that $l' \neq l$,

(iv) raises $\widetilde{\mu}^f$, $\widetilde{s}^f$, s^f, and π^f,

(v) lowers $\widetilde{\mu}^g$, $\widetilde{s}^g$, and π^g for every $g \neq f$ in $\mathcal{F}_l$,

(vi) lowers μ^g (respectively, $\widetilde{\mu}^g$), s^g, and π^g for every $g \in \mathcal{F} \setminus \mathcal{F}_l$.

Similarly, let $f \in \mathcal{F}^b$. In equilibrium, an increase in T^f

(vii) raises H^*,

(viii) raises H_l and lowers s_l for every $l \in \mathcal{L}^n$,

(ix) raises μ^f, s^f, and π^f,

(x) lowers μ^g (respectively, $\widetilde{\mu}^g$), s^g, and π^g for every $g \in \mathcal{F} \setminus \{f\}$.

Proof. See Section VIII.6.2. □

VIII.2 Modeling Mergers

As mentioned at the beginning of this section, we confine attention to two types of mergers: Broad mergers, which are such that the merger partners and the merged firm are all broad firms; and narrow mergers, which are such that the merger partners and the merged firm are all narrow firms operating in the same nest. Regardless of whether a merger is broad or narrow, the type aggregation property implies that the post-merger type T^M is a sufficient statistic for the behavior of the merged firm. We therefore continue to be agnostic on whether a given merger leads to new products being introduced (or old products being withdrawn) by the merged firm, or whether the qualities and unit costs of the merged firms' pre-existing products increase or decrease. We continue to assume that the product portfolios of non-merging firms are unaffected by the merger.

As in Section 2.4, a broad merger M between the firms in $\mathcal{M}$ involves synergies if the post-merger type satisfies

$$T^M > \sum_{f \in \mathcal{M}} T^f.$$

The definition of types for narrow firms in Section VIII.1 implies that a narrow merger M between the firms in $\mathcal{M}$ involves synergies if

$$(T^M)^{\frac{1}{\beta}} > \sum_{f \in \mathcal{M}} (T^f)^{\frac{1}{\beta}}.$$

VIII.3 Consumer Surplus Effects of Mergers: Static Analysis

Since the behavior of broad firms is still driven by the fitting-in functions $S(\cdot)$, $m(\cdot)$, and $\pi(\cdot)$, the analysis in Section 4.1 applies to broad mergers. Specifically, Propositions III–9 all apply to broad mergers. The objective of this subsection is to prove the analogues of those propositions for narrow mergers.

VIII.3.1 Existence of the cutoff type

Consider a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$ and let T^M denote the post-merger type of the merged firm. Let H^* denote the pre-merger equilibrium aggregator level. The pre-merger equilibrium level of the nest-l aggregator is denoted H_l^*. Suppose that T^M satisfies

$$\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right) = \sum_{f \in \mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right). \tag{xxiii}$$

Then, at H^*, nest l continues to deliver a contribution to the industry aggregator of $(H_l^*)^\beta$. As other nests are not directly affected by the merger, they continue to provide their pre-merger contribution to the industry aggregator. It follows that industry-level market shares continue to add up to unity. Therefore, H^* continues to be the equilibrium aggregator level and merger M is CS-neutral.

The fact that $(0,1) \subseteq \widetilde{S}(\mathbb{R}_{++}, s_l)$ and that $\widetilde{S}(\cdot, s_l)$ is continuous and strictly increasing implies that equation (xxiii) has a unique solution in T^M, denoted $\widehat{T}^M(H_l^*, H^*)$.

If T^M is strictly greater than $\widehat{T}^M$, then, by Proposition II, the post-merger equilibrium aggregator level strictly exceeds H^* and the merger is CS-increasing. The same argument implies that the merger is CS-decreasing if $T^M < \widehat{T}^M$.

We now show that $\left(\widehat{T}^M\right)^{\frac{1}{\beta}} > \sum_{f \in \mathcal{M}} (T^f)^{\frac{1}{\beta}}$. Since $\widetilde{S}$ is strictly sub-additive in its first argument (see Lemma XXVI in Section VIII.6.1), we have that

$$\widetilde{S}\left(\sum_{f \in \mathcal{M}} \frac{(T^f)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right) < \sum_{f \in \mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l^*}, \frac{(H_l^*)^\beta}{H^*}\right).$$

Thus, at $T^M = \left(\sum_{f \in \mathcal{M}} (T^f)^{\frac{1}{\beta}}\right)^\beta$, the left-hand side of equation (xxiii) is strictly lower than the right-hand side. Since that left-hand side is strictly increasing in T^M, it follows that $\left(\widehat{T}^M\right)^{\frac{1}{\beta}} > \sum_{f \in \mathcal{M}} (T^f)^{\frac{1}{\beta}}$. In other words, a CS-nondecreasing merger must involve synergies.

We summarize these insights in the following proposition, which is the analogue of Proposition 7:

Proposition III. *For a narrow merger among the firms in $\mathcal{M} \subseteq \mathcal{F}_l$, there exists a unique*

$$\widehat{T}^M > \left(\sum_{f \in \mathcal{M}} (T^f)^{\frac{1}{\beta}} \right)^{\beta}$$

such that the merger is CS-neutral if the post-merger type satisfies $T^M = \widehat{T}^M$, CS-decreasing if $T^M < \widehat{T}^M$, and CS-increasing if $T^M > \widehat{T}^M$.

VIII.3.2 Impact of the intensity of competition on the cutoff type

Our goal is to prove the analogue of Proposition 8 for narrow mergers. That is, we want to show that a narrow merger requires fewer synergies to be CS-neutral if the merging firms operate in a more competitive environment. Compared to what we do in Section 4.1, the difference is that from the point of view of firm $f \in l$, the intensity of competition is now captured by two proxies: H_l^* and H^*. (Recall from Lemma XXV in Section VIII.6.1 that firm f does perceive an increase in H_l^* as competition becoming more intense since $\widetilde{m}\left(T^f/H_l^*, (H_l^*)^\beta/H^*\right)$ is decreasing in H_l^*.)

We first show that the cutoff type $\widehat{T}^M(H_l, H^*)$ decreases with H^*:

Proposition IV. *For a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$, the cutoff type $\widehat{T}^M\left(H_l^*, H^*\right)$ is strictly decreasing in H^*.*

Proof. See Section VIII.6.3. □

Next, we show that the cutoff type $\widehat{T}^M(H_l^*, H^*)$ decreases with H_l^*:

Proposition V. *For a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$, the cutoff type $\widehat{T}^M\left(H_l^*, H^*\right)$ is strictly decreasing in H_l^*.*

Proof. See Section VIII.6.4. □

A final thought experiment is to raise H_l^* while holding fixed $(H_l^*)^\beta/H^*$, the market share of nest l:

Proposition VI. *Consider a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$. For every $s_l \in (0,1)$,*

$$\left. \frac{\partial \widehat{T}^M(H_l^*, H^*)}{\partial H_l^*} \right|_{(H_l^*)^\beta/H^* = s_l} < 0.$$

Proof. This boils down to showing that $H_l \mapsto \widehat{T}^M\left(H_l, H_l^\beta/s_l\right)$ is strictly decreasing, which holds true by Propositions IV and V. □

VIII.3.3 Impact of pre-merger types on the cutoff type

We now prove the analogue of Proposition 9 for narrow mergers. That is, we show that narrow mergers involving larger firms require larger synergies to be CS-nondecreasing, holding fixed the pre-merger aggregator levels:

Proposition VII. *Consider a narrow merger between the firms in $\mathcal{M} = \{f, g\} \subseteq \mathcal{F}_l$, resp., $\mathcal{M}' = \{f', g'\} \subseteq \mathcal{F}_l$, where $T^f \geq T^{f'}$ and $T^g > T^{g'}$. Then, the "larger" merger $\mathcal{M}$ requires larger synergies than $\mathcal{M}'$, in the sense of a larger fractional increase in type:*

$$\frac{(\widehat{T}^M)^{\frac{1}{\beta}}}{(T^f)^{\frac{1}{\beta}} + (T^g)^{\frac{1}{\beta}}} > \frac{(\widehat{T}^{M'})^{\frac{1}{\beta}}}{(T^{f'})^{\frac{1}{\beta}} + (T^{g'})^{\frac{1}{\beta}}}.$$

This in turn implies that the larger merger requires a larger absolute increase in type:

$$(\widehat{T}^M)^{\frac{1}{\beta}} - \left((T^f)^{\frac{1}{\beta}} + (T^g)^{\frac{1}{\beta}}\right) > (\widehat{T}^{M'})^{\frac{1}{\beta}} - \left((T^{f'})^{\frac{1}{\beta}} + (T^{g'})^{\frac{1}{\beta}}\right).$$

Proof. The argument in the proof of Proposition 9 relies solely on the following properties of $S(\cdot)$: $S(\cdot)$ is smooth, positive, concave, strictly increasing, and sub-additive; $\varepsilon(\cdot)$, the elasticity of $S(\cdot)$, is strictly decreasing; $S''(\cdot)/(S'(\cdot))^2$ is strictly decreasing. Since $\widetilde{S}(\cdot, s_l)$ satisfies the same properties (see Lemmas XXVI and XXIX in Section VIII.6.1) for every s_l and since the argument s_l does not vary in the statement of the proposition, the same argument can be applied to obtain the proposition. □

VIII.4 Consumer Surplus Effects of Mergers: Dynamic Analysis

The dynamic framework is the same as in Section 4.2. To ensure that the results derived in Sections 4 and VIII.3 apply, we assume that every merger is either broad or narrow.

As in the main text, our goal is to establish the dynamic optimality of a CS-maximizing merger policy. The analysis proceeds in two main steps. First, we show that the myopically CS-maximizing merger policy maximizes discounted consumer surplus *if all feasible but not yet approved mergers are proposed in each period.* Second, we show that there exists a subgame-perfect equilibrium in which all feasible but not yet approved mergers are indeed proposed in each period. Moreover, any subgame-perfect equilibrium induces the same optimal sequence of period-by-period consumer surpluses.

We begin with the following observation:

Lemma XXII. *Suppose a broad or narrow merger takes place. If the merger is CS-increasing (resp. CS-decreasing), then H^* and H_l^* increase (resp. decrease) for every nest $l \in \mathcal{L}^n$. If it is CS-neutral, then H^* and H_l^* remain constant for every nest $l \in \mathcal{L}^n$.*

Proof. Consider a broad or narrow merger M between the firms in $\mathcal{M}$. If $T^M = \widehat{T}^M$, then, by definition of the cutoff type, the merger is CS-neutral and the merger affects neither the industry aggregator nor the nest-level aggregators. If T^M increases above $\widehat{T}^M$, then the industry aggregator and the nest-level aggregators increase strictly by Proposition II. Therefore, a CS-increasing merger raises all aggregators strictly. The same argument implies that a CS-decreasing merger lowers all aggregators strictly. □

Next, using Lemma XXII and Propositions 8, IV, and V, we establish the sign-preserving complementarity between CS-nondecreasing (resp. CS-nonincreasing) broad or narrow mergers, extending Proposition 10:

Proposition VIII. *If broad or narrow merger M_k is CS-nondecreasing in isolation, it remains CS-nondecreasing if another broad or narrow merger $M_{k'}$, $k' \neq k$, that is CS-nondecreasing in isolation takes place. If broad or narrow merger M_k is CS-decreasing in isolation, it remains CS-decreasing if another broad or narrow merger $M_{k'}$, $k' \neq k$, that is CS-decreasing in isolation takes place.*

Proof. A CS-nondecreasing merger weakly raises the industry aggregator and all nest aggregators by Lemma XXII. Hence, such a merger weakly lowers the cutoff types of all other mergers by Propositions 8, IV, and V. The same argument implies that a CS-nonincreasing merger weakly raises the cutoff types of all other mergers. □

Proposition 11 extends trivially:

Proposition IX. *Suppose that broad or narrow merger M_k is CS-nondecreasing in isolation whereas broad or narrow merger $M_{k'}$ is CS-decreasing in isolation but CS-nondecreasing once merger M_k has taken place. Then, merger M_k is CS-increasing conditional on merger $M_{k'}$ taking place.*

Combining the argument in Lemma 4 in Nocke and Whinston (2010) and Propositions VIII and IX, we obtain the following corollary:

Corollary I. *Suppose that all feasible but not yet approved mergers are proposed in each period. Then, the myopically CS-maximizing merger policy maximizes discounted consumer surplus, no matter what the realization of feasible mergers is.*

Next, we turn to the second part of our analysis. That is, we show that there always exists a subgame-perfect equilibrium in which, in each period, every feasible but not yet approved merger is proposed for approval.

We begin by showing that a CS-nondecreasing merger is privately profitable, extending Proposition 12:

Proposition X. *A CS-nondecreasing broad or narrow mergers is privately profitable in that it strictly raises the joint profit of the merger partners, holding fixed the market structure among outsiders.*

Proof. See Section VIII.6.5. □

The second step is to extend Proposition 13. That is, we want to show that a CS-nondecreasing merger is still privately profitable even if it induces (directly or indirectly) other mergers to become CS-nondecreasing, resulting in their approval. The following observation will be useful:

Lemma XXIII. *A CS-increasing (resp. CS-decreasing) broad or narrow merger lowers (resp. raises) the equilibrium profits of every outsider. A CS-neutral merger does not affect outsiders' profits.*

Proof. Consider a broad or narrow merger M between the firms in $\mathcal{M}$. Suppose that the post-merger type T^M is equal to $\widehat{T}^M$, so that merger M is CS-neutral. As the merger affects none of the aggregators by Lemma XXII, it has no impact on the profits made by rival firms. Starting from this outcome, a CS-increasing merger is formally equivalent to increasing the post-merger type above $\widehat{T}^M$. We know from Proposition II that this results in lower profits for all rivals. The same argument implies that a CS-decreasing merger raises the profits of every outsider. □

We obtain the following proposition:

Proposition XI. *Suppose that broad or narrow merger M_k is CS-nondecreasing given current market structure whereas broad or narrow merger $M_{k'}$ is CS-decreasing but becomes CS-nondecreasing once M_k has been implemented. Then, the joint profit of the firms in $\mathcal{M}_k$ is strictly higher if both mergers take place than if none does.*

Proof. Given Lemma XXIII and Propositions IX and X, the argument is exactly the same as in the proof of Proposition 13. □

Combining the results shown above with a backward induction argument, we obtain the main result of this section, extending Proposition 14:

Proposition XII. *Suppose that the antitrust authority adopts the myopically CS-maximizing merger policy. Then, all feasible mergers being proposed in each period after any history is a subgame-perfect equilibrium. The resulting outcome maximizes discounted consumer surplus, no matter what the realized sequence of feasible mergers. Moreover, every subgame-perfect equilibrium results in the same optimal level of consumer surplus in each period.*

VIII.5 Comparing Broad and Narrow Mergers

Fix a vector of industry-level market shares $(s^f)_{f\in\mathcal{M}}$, where $\mathcal{M}$ is a finite set containing at least two elements. In this subsection, we study whether a merger between the firms in $\mathcal{M}$ requires more or fewer synergies to be CS-neutral, depending on whether the merger is broad or narrow. Let $\overline{s} = \sum_{f\in\mathcal{M}} s^f$ be the combined industry-level market share of the merger partners. If the merger is narrow, let $s_l \geq \overline{s}$ be the market share of the nest where the merger partners are operating.

In Section VIII.5.1, we compare the required synergy level for broad and narrow mergers under NMNL demand, allowing merger partners to be asymmetric (Proposition XIII). In Section VIII.5.2, we do so under NCES demand, confining attention to the case where merger partners are symmetric (Lemma XXIV).

VIII.5.1 The case of NMNL demand

Suppose first that merger $\mathcal{M}$ is a broad merger. The cutoff type that makes this merger CS-neutral satisfies

$$\overline{s} = \frac{T_b^M}{H} \exp\left(-\frac{1}{1-\overline{s}}\right),$$

where we have used equations (10) and (11). It follows that

$$\frac{T_b^M}{H} = \overline{s} \exp \frac{1}{1-\overline{s}}.$$

Similarly, the pre-merger type of firm $f \in \mathcal{M}$ satisfies

$$\frac{T_b^f}{H} = s^f \exp \frac{1}{1-s^f}.$$

The required synergy level for the broad merger is therefore given by

$$E_b = \frac{T_b^M}{\sum_{f\in\mathcal{M}} T_b^f} = \frac{\overline{s} \exp \frac{1}{1-\overline{s}}}{\sum_{f\in\mathcal{M}} s^f \exp \frac{1}{1-s^f}}.$$

Next, suppose instead that the merger is narrow. The cutoff type satisfies

$$\frac{\overline{s}}{s_l} = \frac{(T_n^M)^{\frac{1}{\beta}}}{H_l} \exp\left(-\frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right),$$

where we have used equations (16) and (17). It follows that

$$\frac{(T_n^M)^{\frac{1}{\beta}}}{H_l} = \frac{\overline{s}}{s_l} \exp \frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}.$$

Similarly, the pre-merger type of firm $f \in \mathcal{M}$ satisfies

$$\frac{(T_n^f)^{\frac{1}{\beta}}}{H_l} = \frac{s^f}{s_l} \exp \frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}}.$$

This pins down the required synergy level as

$$E_n = \frac{T_n^M}{\left(\sum_{f\in\mathcal{M}}(T_n^f)^{\frac{1}{\beta}}\right)^\beta} = \left(\frac{\overline{s}\exp\frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}}{\sum_{f\in\mathcal{M}} s^f \exp\frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}}}\right)^\beta .$$

Comparing E_b and E_n, we obtain:

Proposition XIII. *Consider two equivalent broad and narrow mergers. Let $\overline{s}$ be the combined pre-merger industry-level market shares of the merger partners and s_l the pre-merger market share of the narrow merger's nest. Suppose that demand is of the NMNL type and that $s^f/s_l \leq 3/4$ for every f.*

There exists a threshold $\widehat{s}_l \in (\overline{s}, 1)$ such that the broad merger requires fewer synergies than the narrow one, $E_b < E_n$, if $s_l < \widehat{s}_l$, whereas the opposite is true if $s_l > \widehat{s}_l$.

Proof. See Section VIII.6.6. □

Note that the condition that $s^f/s_l \leq 3/4$ for every f is automatically satisfied if the merger partners are symmetric, implying Proposition 18 in the case of NMNL demand.

VIII.5.2 The case of NCES demand

Suppose the merger partners are symmetric: $s^f = \overline{s}/N$ for every f, where N is the number of merging firms. If merger $\mathcal{M}$ is a broad merger, then the cutoff type satisfies

$$\overline{s} = \frac{T_b^M}{H}\left(1 - \frac{1-\alpha}{1-\alpha\overline{s}}\right)^{\frac{\alpha}{1-\alpha}},$$

i.e.,

$$\frac{T_b^M}{H} = \overline{s}\left(\frac{1}{\alpha}\frac{1-\alpha\overline{s}}{1-\overline{s}}\right)^{\frac{\alpha}{1-\alpha}}.$$

Similarly, the pre-merger type of every merger partner satisfies

$$\frac{T_b}{H} = \frac{\overline{s}}{N}\left(\frac{1}{\alpha}\frac{1-\alpha\frac{\overline{s}}{N}}{1-\frac{\overline{s}}{N}}\right)^{\frac{\alpha}{1-\alpha}}.$$

The required synergy level for the broad merger is therefore given by

$$E_b = \frac{T_b^M}{NT_b} = \left(\frac{1-\alpha\overline{s}}{1-\overline{s}}\frac{1-\frac{\overline{s}}{N}}{1-\alpha\frac{\overline{s}}{N}}\right)^{\frac{\alpha}{1-\alpha}}.$$

Using the fact that $\alpha = \widetilde{\alpha}\beta/(1-\widetilde{\alpha}(1-\beta))$ and simplifying, we obtain:

$$E_b = \left(\frac{1-\widetilde{\alpha}(1-\beta+\beta\overline{s})}{1-\overline{s}}\frac{1-\frac{\overline{s}}{N}}{1-\widetilde{\alpha}\left(1-\beta+\beta\frac{\overline{s}}{N}\right)}\right)^{\frac{\widetilde{\alpha}\beta}{1-\widetilde{\alpha}}}.$$

Next, suppose instead that the merger is narrow. The cutoff type satisfies

$$\frac{\overline{s}}{s_l} = \frac{(T_n^M)^{\frac{1}{\beta}}}{H_l}\left(1 - \frac{1-\widetilde{\alpha}}{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}},$$

i.e.,

$$\frac{(T_n^M)^{\frac{1}{\beta}}}{H_l} = \frac{\overline{s}}{s_l}\left(\frac{1}{\widetilde{\alpha}}\frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}}.$$

Similarly, the pre-merger type of every merger partner satisfies

$$\frac{(T_n)^{\frac{1}{\beta}}}{H_l} = \frac{\overline{s}}{Ns_l}\left(\frac{1}{\widetilde{\alpha}}\frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}\right)^{\frac{\widetilde{\alpha}}{1-\widetilde{\alpha}}}.$$

This pins down the required synergy level for the narrow merger as

$$E_n = \frac{\widetilde{T}^M}{\left(NT_n^{\frac{1}{\beta}}\right)^{\beta}} = \left(\frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\frac{1-(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{Ns_l}}\right)^{\frac{\widetilde{\alpha}\beta}{1-\widetilde{\alpha}}}.$$

Comparing E_b and E_n, we obtain Proposition 18 in the case of NCES demand:

Lemma XXIV. *Consider two equivalent broad and narrow mergers involving symmetric firms. Let $\overline{s}$ be the combined pre-merger industry-level market shares of the merger partners and s_l the pre-merger market share of the narrow merger's nest. Suppose that demand is of the NCES type.*

There exists a threshold $\widehat{s}_l \in (\overline{s}, 1)$ such that the broad merger requires fewer synergies than the narrow one, $E_b < E_n$, if $s_l < \widehat{s}_l$, whereas the opposite is true if $s_l > \widehat{s}_l$.

Proof. See Section VIII.6.7. □

VIII.6 Proofs

VIII.6.1 Preliminary technical lemmas

The following lemma will be used in the proof of Proposition II:

Lemma XXV. *The mapping $H_l \mapsto \widetilde{m}\left((T^f)^{\frac{1}{\beta}}/H_l, H_l^{\beta}/H\right)$ is strictly decreasing.*

Proof. Note that

$$\frac{d}{dH_l}\widetilde{m}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^{\beta}}{H}\right) = \frac{1}{H_l}\left(-\frac{(T^f)^{\frac{1}{\beta}}}{H_l}\partial_1\widetilde{m}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^{\beta}}{H}\right) + \beta\frac{H_l^{\beta}}{H}\partial_2\widetilde{m}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^{\beta}}{H}\right)\right).$$

Thus, all we need to do is show that $\beta y \partial_2 \widetilde{m}(x,y) - x\partial_1 \widetilde{m}(x,y) < 0$. Totally differentiating equations (xvi) and (xvii), we obtain:

$$\beta y \partial_2 \widetilde{m}(x,y) - x\partial_1 \widetilde{m}(x,y) = \frac{-\widetilde{\alpha} s(1-\beta)\left(1-(1-\beta+\beta y)\, s\right)(1+\beta y)}{\left(1-\widetilde{\alpha}\left(1-\beta+\beta y\right) s\right)\left(1-\left(1-\beta+\beta y\right) s+\widetilde{\alpha}\left(1-\beta+\beta y\right)^2 s^2\right)},$$

where $s = \widetilde{S}(x,y)$. The above expression is clearly negative since $(1-\beta+\beta y)\widetilde{S}(x,y) < 1$. □

The following lemma will be used in the proof of Propositions III and VII:

Lemma XXVI. *$\widetilde{S}(\cdot, s_l)$ is strictly concave. Therefore, it is strictly sub-additive.*

Proof. Applying the implicit function theorem to equations (xvi) and (xvii), we obtain:

$$\partial_1 \widetilde{S}(x,y) = \frac{\widetilde{S}(x,y)\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)\right)\left(1-\widetilde{\alpha}(1-\beta+\beta y)\widetilde{S}(x,y)\right)}{x\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x,y)^2\right)}. \tag{xxiv}$$

Differentiating equation (xxiv) with respect to x and plugging in the value of $\partial_1 \widetilde{S}(x,y)$ from that same equation yields:

$$\partial_{11}^2 \widetilde{S}(x,y) = \frac{\widetilde{\alpha} s^2(1-\beta+\beta y)\left(2-(1-\beta+\beta y)s\right)\left(1-(1-\beta+\beta y)s\right)\left(1-\widetilde{\alpha}(1-\beta+\beta y)s\right)}{-x^2\left(1-(1-\beta+\beta y)s+\widetilde{\alpha}(1-\beta+\beta y)^2 s^2\right)^3},$$

where $s = \widetilde{S}(x,y)$. Since $(1-\beta+\beta y)\widetilde{S}(x,y) < 1$, it follows that $\partial_{11}^2 \widetilde{S}(x,y) < 0$ and that $\widetilde{S}(\cdot, y)$ is strictly sub-additive. □

The following lemma will be used in the proof of Proposition IV:

Lemma XXVII. *For every (x,y), $\partial_2 \widetilde{S}(x,y) = -\zeta\left(\widetilde{S}(x,y)\right)$, where*

$$\zeta(s) \equiv \frac{\widetilde{\alpha}\beta s^2}{1-(1-\beta+\beta y)s+\widetilde{\alpha}(1-\beta+\beta y)^2 s^2}.$$

Moreover, ζ is strictly super-additive on the interval $(0, 1/(1-\beta+\beta y))$ for every $(\widetilde{\alpha}, \beta, y) \in (0,1] \times (0,1)^2$.

Proof. The fact that $\partial_2 \widetilde{S}(x,y) = -\zeta\left(\widetilde{S}(x,y)\right)$ follows immediately by applying the implicit function theorem to equations (xvi) and (xvii). Note that establishing the strict super-additivity of ζ on $(0, 1/(1-\beta+\beta y))$ is equivalent to establishing the strict super-additivity of

$$\tilde{\zeta}(s) = \frac{s^2}{1-s+\widetilde{\alpha} s^2}$$

on $(0,1)$ for every $\bar{s} \in (0,1)$, which we undertake next.

Note that $\tilde{\zeta}''(s)$ has the same sign as $P(s) \equiv 1-\widetilde{\alpha}(3-s)s^2$. The polynomial P is strictly decreasing on $[0,1]$. Hence, if $\widetilde{\alpha} \leq 1/2$, then $P(s) > 0$ for every $s \in [0,1)$ and $\tilde{\zeta}$ is strictly convex on $(0,1)$. If instead $\widetilde{\alpha} > 1/2$, then $P(1) < 0$ and there exists a unique $\hat{s}(\widetilde{\alpha}) \in (0,1)$ such that $P(\hat{s}(\widetilde{\alpha})) = 0$. It is easy to check that $P(1/2) > 0$, so that $\hat{s}(\widetilde{\alpha}) > 1/2$ for every $\widetilde{\alpha} > 1/2$.

To sum up, if $\widetilde{\alpha} \leq 1/2$, then $\tilde{\zeta}$ is strictly convex and hence strictly super-additive on $(0,1)$. If instead $\widetilde{\alpha} > 1/2$, which we assume in the following, then $\tilde{\zeta}$ is strictly convex on $(0,\hat{s}(\widetilde{\alpha}))$ and strictly concave on $(\hat{s}(\widetilde{\alpha}),1)$. We want to show that, for every $n \geq 2$, $s \in (0,1)$, and $(s_i)_{1\leq i\leq n} \in (0,s)^n$ such that $\sum_{i=1}^n s_i = s$, $\sum_{i=1}^n \tilde{\zeta}(s_i) < \tilde{\zeta}(s)$.

Let $s \in (0,1)$. Define

$$\mathcal{S}^n = \left\{ (s_i)_{1\leq i\leq n} \in [0,s]^n : \sum_{i=1}^n s_i = s \right\}, \quad \forall n \geq 1,$$
$$\text{and } \mathcal{S} = \bigcup_{n\geq 1} \mathcal{S}^n,$$

and consider the following maximization problem:

$$\max_{(s_i)_{1\leq i\leq n}\in\mathcal{S}} \sum_{i=1}^n \tilde{\zeta}(s_i). \tag{xxv}$$

We need to show that $(s_i)_{1\leq i\leq n} \in \mathcal{S}$ solves the above maximization problem if and only if $s_i = s$ for some i.

An induction argument similar to the one in the proof of Proposition 16 implies that, for every $(s_i)_{1\leq i\leq n} \in \mathcal{S}$ with at least three strictly positive components, there exists $(s'_i)_{1\leq i\leq 2} \in \mathcal{S}^2$ such that $\sum_{i=1}^n \tilde{\zeta}(s_i) < \tilde{\zeta}(s'_1)+\tilde{\zeta}(s'_2)$. Since, by continuity and compactness, the maximization problem

$$\max_{(s_1,s_2)\in\mathcal{S}^2} \tilde{\zeta}(s_1) + \tilde{\zeta}(s_2) \tag{xxvi}$$

has a solution, we can conclude that maximization problem (xxv) has a solution, and that any solution has at most two strictly positive components. All we need to do now is show that maximization problem (xxvi) has exactly two solutions: $(s,0)$ and $(0,s)$. This boils down to showing that $\arg\max_{\lambda\in[0,s]} \chi(\lambda) = \{0,s\}$, where $\chi(\lambda) = \tilde{\zeta}(\lambda) + \tilde{\zeta}(s-\lambda)$. Routine but tedious calculations show that this is the case as χ is strictly decreasing on $(0,s/2)$ and strictly increasing on $(s/2,s)$. □

The following lemma will be used in the proof of Proposition V:

Lemma XXVIII. *For every* (x,y), $-x\partial_1\widetilde{S}(x,y) + \beta y\partial_2\widetilde{S}(x,y) = \phi\left(\widetilde{S}(x,y)\right)$, *where*

$$\phi(s) \equiv -s + \frac{1-\beta}{\beta}(1+\beta y)\zeta(s),$$

where ζ was defined in Lemma XXVII. Moreover, ϕ is strictly super-additive on the interval $(0, 1/(1-\beta+\beta y))$ for every $(\widetilde{\alpha}, \beta, y) \in (0,1] \times (0,1)^2$.

Proof. The fact that $-x\partial_1 \widetilde{S}(x,y) + \beta y \partial_2 \widetilde{S}(x,y) = \phi\left(\widetilde{S}(x,y)\right)$ follows immediately by applying the implicit function theorem to equations (xvi) and (xvii). The strict super-additivity of ϕ follows as ϕ is the sum of a linear function and a strictly super-additive function (recall Lemma XXVII). □

The following lemma will be used in the proof of Proposition VII

Lemma XXIX. *For every (x,y), let $\widetilde{\varepsilon}(x,y) = x\partial_1 \widetilde{S}(x,y)/\widetilde{S}(x,y)$. Then, $\widetilde{\varepsilon}(\cdot, y)$ is strictly decreasing. Moreover, $\partial_{11}\widetilde{S}(\cdot,y)/\left(\partial_1 \widetilde{S}(\cdot,y)\right)^2$ is strictly decreasing.*

Proof. Equation (xxiv) implies that

$$\widetilde{\varepsilon}(x,y) = \frac{\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)\right)\left(1-\widetilde{\alpha}(1-\beta+\beta y)\widetilde{S}(x,y)\right)}{1-(1-\beta+\beta y)\widetilde{S}(x,y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x,y)^2}.$$

Differentiating this with respect to x yields:

$$\partial_1 \widetilde{\varepsilon}(x,y) = \frac{-\widetilde{\alpha}(1-\beta+\beta y)\left(1-\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x,y)^2\right)}{\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x,y)^2\right)^2}\partial_1 \widetilde{S}(x,y),$$

which is strictly negative.

Using equation (xxiv), we also obtain an expression for $\partial_{11}\widetilde{S}(x,y)/\left(\partial_1 \widetilde{S}(x,y)\right)^2$:

$$\begin{aligned}\frac{\partial_{11}^2 \widetilde{S}(x,y)}{(\partial_1 \widetilde{S}(x,y))^2} &= -\widetilde{\alpha}\left(1-\beta+\beta y\right)\left(2-(1-\beta+\beta y)\widetilde{S}(x,y)\right)/\left(\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)\right)\right.\\ &\quad \left.\times\left(1-\widetilde{\alpha}(1-\beta+\beta y)\widetilde{S}(x,y)\right)\left(1-(1-\beta+\beta y)\widetilde{S}(x,y)+\widetilde{\alpha}(1-\beta+\beta y)^2\widetilde{S}(x,y)^2\right)\right),\end{aligned}$$

which is strictly negative. □

VIII.6.2 Proof of Proposition II

Proof. Let $l \in \mathcal{L}^n$ and $f \in \mathcal{F}_l$. In this proof, we make explicit the dependence of the fitting-in functions $H_{l'}(\cdot)$ and $\Sigma_{l'}(\cdot)$ on T^f by writing $H_{l'}(\cdot, T^f)$ and $\Sigma_{l'}(\cdot, T^f)$ for every $l' \in \mathcal{L}^n$. The equilibrium values of all variables are denoted with a star superscript.

Since $\widetilde{S}$ is increasing in the first argument and decreasing in the second argument, an increase in T^f raises the left-hand side of equation (xviii). Since that left-hand side is strictly decreasing in H_l, it follows that $H_l(H, T^f)$ is strictly increasing in T^f. Hence, $\Sigma_l(H, T^f)$ is strictly increasing in T^f. By contrast, the increase in T^f leaves the left-hand side of equation (xviii) unaffected for nests $l' \neq l$. It follows that for $l' \neq l$, $H_{l'}(H, T^f)$ and $\Sigma_{l'}(H, T^f)$ are both constant in T^f. This immediately implies that the left-hand side of equation (xx) increases as T^f increases. Since that left-hand side is strictly decreasing in H, it follows that H^* is strictly increasing in T^f, which proves part (i) of the proposition.

Let $l' \in \mathcal{L}^n$ such that $l' \neq l$. Since $\Sigma_{l'}(H, T^f)$ decreases with H but is constant in T^f, we immediately have that $s^*_{l'}$ is strictly decreasing in T^f. Similarly, since $H_{l'}(H, T^f)$ increases with H but does not depend on T^f, we also have that $H^*_{l'}$ is strictly increasing in T^f. This proves part (iii) of the proposition.

Since the fitting-in functions $m(\cdot)$, $S(\cdot)$, and $\pi(\cdot)$ are all decreasing, and since H^* is increasing in T^f, it follows that μ^{*g}, s^{*g}, and π^{*g} are all strictly decreasing in T^f for every $g \in \mathcal{F}^b$. Next, let $l' \in \mathcal{L}^n \setminus \{l\}$ and $g \in l'$. Firm g's equilibrium ι-markup $\widetilde{\mu}^{*g}$ is given by $\widetilde{m}((T^g)^{\frac{1}{\beta}}/H^*_{l'}, s^*_{l'})$. Since $\widetilde{m}$ is increasing in both arguments, and since both arguments are decreasing in T^f, it follows that $\widetilde{\mu}^{*g}$ is strictly decreasing in T^f. Combining this with the fact that $s^*_{l'}$ is decreasing in T^f and using equation (xxii) allows us to conclude that π^{*g} is strictly decreasing in T^f. Assume for a contradiction that $s^{*g}(T^{f\prime}) \geq s^{*g}(T^f)$ for some $T^{f\prime} > T^f$. Since $s^*_{l'}(T^{f\prime}) < s^*_{l'}(T^f)$, it must be the case that $\widetilde{s}^{*g}(T^{f\prime}) > \widetilde{s}^{*g}(T^f)$. Using equation (xvi), this implies that $\widetilde{\mu}^{*g}(T^{f\prime}) > \widetilde{\mu}^{*g}(T^f)$, which contradicts the fact that $\widetilde{\mu}^{*g}$ is strictly decreasing in T^f. Hence, s^{*g} is strictly decreasing in T^f. This proves part (vi) of the proposition.

We have shown above that $s^*_{l'}$ is strictly decreasing in T^f for every $l' \in \mathcal{L}^n \setminus \{l\}$ and that s^{*g} is strictly decreasing in T^f for every $g \in \mathcal{F}^b$. Since the equilibrium market share of the outside option, H^0/H^*, is non-increasing in T^f and since market shares must add up to unity, it follows that s^*_l is strictly increasing in T^f. Moreover, as $H_l(H, T^f)$ is strictly increasing in both of its arguments, we also have that H^*_l is strictly increasing in T^f, which proves part (ii) of the proposition.

Let $g \in \mathcal{F}_l$ such that $g \neq f$. Firm g's equilibrium market share in nest l, $\widetilde{s}^{*g}$, is given by $\widetilde{S}((T^g)^{\frac{1}{\beta}}/H^*_l, s^*_l)$. Since $\widetilde{S}$ is strictly increasing in its first argument and strictly decreasing in its second argument, and since $(T^g)^{\frac{1}{\beta}}/H^*_l$ and s^*_l are respectively strictly decreasing and strictly increasing in T^f, it follows that $\widetilde{s}^{*g}$ is strictly decreasing in T^f. Moreover, as market shares within nest l add up to unity, we have that $\widetilde{s}^{*f}$ is increasing in T^f. Combining this with the fact that s^*_l is increasing in T^f and using equations (xvi) and (xxii) allows us to conclude that s^{*f}, $\widetilde{\mu}^{*f}$, and π^{*f} are all strictly increasing in T^f, which proves part (iv) of the proposition.

Finally, we prove part (v) of the proposition. Let $g \in l$ such that $l \neq f$. We have already shown that $\widetilde{s}^{*g}$ is strictly decreasing in T^f. Firm g's equilibrium ι-markup, μ^{*g}, is given by $\widetilde{m}\left((T^g)^{\frac{1}{\beta}}/H^*_l, (H^*_l)^\beta/H^*\right)$. This expression is strictly decreasing in H^*_l (by Lemma XXV) and H^* (since $\widetilde{m}$ is strictly increasing in its second argument). Since H^*_l and H^* are both increasing in T^f, it follows that $\widetilde{\mu}^{*g}$ is strictly decreasing in T^f. To show that π^{*g} is strictly decreasing in T^f, define

$$H^0_{-l}(T^f) = H^0 + \sum_{l' \in \mathcal{L}^n \setminus \{l\}} \left(H^*_{l'}(T^f)\right)^\beta + \sum_{f' \in \mathcal{F}^b} H^*(T^f) S\left(\frac{T^{f'}}{H^*}\right)$$

and

$$H^0_{-g}(T^f) = \sum_{f' \in \mathcal{F}_l \setminus \{g\}} \left(T^{f'}\right)^{\frac{1}{\beta}} \psi\left(\widetilde{\mu}^{*f'}(T^f)\right).$$

We have shown that the second term in the definition of $H^0_{-l}(T^f)$, which represents the contribution of nests $l' \in \mathcal{L}^n \setminus \{l\}$ to the industry aggregator, increases with T^f. Similarly, the third term in the definition of $H^0_{-l}(T^f)$, which represents the contribution of broad firms to the industry aggregator, is also increasing in T^f since broad firms lower their ι-markups as T^f increases. Hence, $H^0_{-n}(T^f)$ is strictly increasing in T^f.

Next, we argue that $H^0_{-g}(T^f)$, the contribution of firm g's rivals to the nest aggregator H_l, also increases with T^f. Clearly, $(T^{f'})^{\frac{1}{\beta}}\psi\left(\widetilde{\mu}^{*f'}(T^f)\right)$ is increasing in T^f for every $f' \neq f$ since $\widetilde{\mu}^{*f'}$ is strictly decreasing in T^f. Moreover, we also have that $(T^f)^{\frac{1}{\beta}}\psi\left(\widetilde{\mu}^{*f}(T^f)\right) = H^*_l(T^f)\widetilde{s}^{*f}(T^f)$ increases with T^f since H^*_l and $\widetilde{s}^{*f}$ are both strictly increasing in T^f. Therefore, $H^0_{-g}(T^f)$ is strictly increasing in T^f.

The profit of firm g when it sets a ι-markup of $\widetilde{\mu}^g$ is given by:

$$\Pi(\widetilde{\mu}^g, T^f) = \alpha\beta(T^g)^{\frac{1}{\beta}}\widetilde{\mu}^g\psi(\widetilde{\mu}^g)\frac{\left(H^0_{-g}(T^f) + (T^g)^{\frac{1}{\beta}}\psi(\widetilde{\mu}^g)\right)^{\beta-1}}{\left(H^0_{-g}(T^f) + (T^g)^{\frac{1}{\beta}}\psi(\widetilde{\mu}^g)\right)^{\beta} + H^0_{-l}(T^f)},$$

which is strictly decreasing in $H^0_{-g}(T^f)$ and $H^0_{-l}(T^f)$ and thus strictly decreasing in T^f. Combining this with a standard revealed profitability argument, we can conclude that firm g's equilibrium profits fall as T^f rises. This proves part (v) of the proposition.

The proof of parts (vii)–(x) is analogous and therefore omitted. □

VIII.6.3 Proof of Proposition IV

Proof. Fix H_l and let

$$\xi(T^M, H) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) - \sum_{f \in \mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right).$$

Note that $\xi(T^M, H) = 0$ if and only if $T^M = \widehat{T}^M(H_l, H)$. Moreover, $\partial_1\xi > 0$. All we need to do is show that $\partial_2\xi(T^M, H) > 0$ whenever $\xi(T^M, H) = 0$. The proposition will then follow by the implicit function theorem.

Let (T^M, H) such that $\xi(T^M, H) = 0$. Note that

$$\begin{aligned}\partial_2\xi(T^M, H) &= \frac{H_l^\beta}{H^2}\left(\sum_{f \in \mathcal{M}} \partial_2\widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) - \partial_2\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right), \\ &= \frac{H_l^\beta}{H^2}\left(\zeta\left(\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right) - \sum_{f \in \mathcal{M}} \zeta\left(\widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right)\right),\end{aligned}$$

$$> 0,$$

where we have used the function ζ defined in Lemma XXVII and the inequality follows from the fact that ζ is super-additive (Lemma XXVII) and

$$\sum_{f\in\mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)$$

as $\xi(T^M, H) = 0$. □

VIII.6.4 Proof of Proposition V

Proof. The approach is similar to the one used in the proof of Proposition IV. Fix H, and define

$$\xi(T^M, H_l) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) - \sum_{f\in\mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right).$$

Note that $\xi(T^M, H_l) = 0$ if and only if $T^M = \widehat{T}^M(H_l, H)$, and that $\partial_1\xi > 0$. We need to show that $\partial_2\xi(T^M, H_l) > 0$ whenever $\xi(T^M, H_l) = 0$.

Let (T^M, H_l) such that $\xi(T^M, H_l) = 0$. Note that

$$\begin{aligned}\partial_2\xi(T^M, H_l) = \frac{1}{H_l}\Bigg(&-\frac{(T^M)^{\frac{1}{\beta}}}{H_l}\partial_1\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) + \beta\frac{H_l^\beta}{H}\partial_2\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\\ &- \sum_{f\in\mathcal{M}}\left(-\frac{(T^f)^{\frac{1}{\beta}}}{H_l}\partial_1\widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) + \beta\frac{H_l^\beta}{H}\partial_2\widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right)\Bigg).\end{aligned}$$

Hence, using the function ϕ defined in Lemma XXVIII,

$$\partial_2\xi(T^M, H_l) = \frac{1}{H_l}\left(\phi\left(\widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right) - \sum_{f\in\mathcal{M}}\phi\left(\widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right)\right)\right) > 0,$$

where the inequality follows from the fact that ϕ is super-additive (Lemma XXVIII) and

$$\sum_{f\in\mathcal{M}} \widetilde{S}\left(\frac{(T^f)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right) = \widetilde{S}\left(\frac{(T^M)^{\frac{1}{\beta}}}{H_l}, \frac{H_l^\beta}{H}\right).$$ □

VIII.6.5 Proof of Proposition X

Proof. The fact that a CS-nondecreasing broad merger is privately profitable follows immediately from the argument in the proof of Proposition 12.

Consider next a narrow merger M between the firms in $\mathcal{M} \subseteq \mathcal{F}_l$. We first show that the merger is profitable if it is CS-neutral, i.e., if $T^M = \widehat{T}^M$. Since $T^M > T^f$ for every

$f \in \mathcal{M}$, and since the merger affects neither H_l^* nor H^*, we have that firm M's ι-markup strictly exceeds the pre-merger ι-markup of every merger partner, i.e., $\widetilde{\mu}^M > \widetilde{\mu}^f$ for every f in $\mathcal{M}$. Moreover, since the merger affects neither H_l^* nor H^*, it does not affect s_l. Finally, by definition of $\widehat{T}^M$, we also have that the market share of firm M in nest l is equal to the sum of the pre-merger market shares of the merger partners, i.e., $\widetilde{s}^M = \sum_{f \in \mathcal{M}} \widetilde{s}^f$. It follows that

$$\begin{aligned} \pi^M &= \widetilde{\alpha}\beta\widetilde{\mu}^M\widetilde{s}^M s_l, \\ &= \widetilde{\alpha}\beta \left(\sum_{f \in \mathcal{M}} \widetilde{\mu}^M \widetilde{s}^f \right) s_l, \\ &> \widetilde{\alpha}\beta \left(\sum_{f \in \mathcal{M}} \widetilde{\mu}^f \widetilde{s}^f \right) s_l, \\ &= \sum_{f \in \mathcal{M}} \pi^f. \end{aligned}$$

The merger is therefore profitable. Moreover, by Proposition II, firm M makes even more profits if its type is $T^M > \widehat{T}^M$. It follows that a CS-nondecreasing merger is profitable. □

VIII.6.6 Proof of Proposition XIII

Proof. The proof proceeds in several steps. We first show that $E_b < E_n$ when $s_l = \overline{s}$. Next, we show that $E_b > E_n$ when $s_l = 1$. Finally, we show that E_b/E_n is strictly increasing in s_l.

Assume that $s_l = \overline{s}$ and define

$$\Psi(\beta) = \beta \log \sum_{f \in \mathcal{M}} \widetilde{s}^f \exp \frac{1}{1 - (1 - \beta + \beta\overline{s})\widetilde{s}^f},$$

where $\widetilde{s}^f = s^f/\overline{s}$. Note that $E_n/E_b = \exp\left(\Psi(1) - \Psi(\beta)\right)$, so all we need to do is show that Ψ is increasing. Differentiating Ψ with respect to β yields:

$$\Psi'(\beta) = \frac{\Psi(\beta)}{\beta} - \frac{\sum_{f \in \mathcal{M}} \frac{\beta\widetilde{s}^f(1-\overline{s})}{\left(1-(1-\beta+\beta\overline{s})\widetilde{s}^f\right)^2} \widetilde{s}^f \exp \frac{1}{1-(1-\beta+\beta\overline{s})\widetilde{s}^f}}{\sum_{f \in \mathcal{M}} \widetilde{s}^f \exp \frac{1}{1-(1-\beta+\beta\overline{s})\widetilde{s}^f}}.$$

Since $\Psi(\beta)/\beta > 1$, a sufficient condition for $\Psi'(\beta)$ to be positive is that

$$\frac{\beta\widetilde{s}^f(1-\overline{s})}{\left(1-(1-\beta+\beta\overline{s})\widetilde{s}^f\right)^2} \leq 1$$

for every f. This condition is equivalent to $\widetilde{s}^f$ being no greater than $\check{s}(\overline{s}, \beta)$ for every f, where

$$\check{s}(\overline{s},\beta) = \frac{2-\beta(1-\overline{s}) - \sqrt{\beta(1-\overline{s})\left(4-3\beta(1-\overline{s})\right)}}{2\left(1-\beta(1-\overline{s})\right)^2}.$$

We find that $\check{s}$ achieves its global minimum at $\overline{s} = 5/16$ and $\beta = 16/33$. The minimized value of $\check{s}$ is $3/4$. Hence, $\Psi'(\beta) > 0$ provided $\widetilde{s}^f \leq 3/4$ for every f. This concludes the first step of the proof.

Next, assume that $s_l = 1$. Then, E_n simplifies to

$$E_n = \left(\frac{\overline{s}\exp\frac{1}{1-\overline{s}}}{\sum_{f\in\mathcal{M}} s^f \exp\frac{1}{1-s^f}}\right)^{\beta} = E_b^{\beta}.$$

Since $E_b > 1$ and $\beta < 1$, it follows that $E_n < E_b$, which concludes the second step of the proof.

For the third step of the proof, note that for every $s_l \in (\overline{s}, 1)$, E_n can be rewritten as

$$E_n = \left(\sum_{f\in\mathcal{M}} \frac{s^f}{\overline{s}} \exp\left(\frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}} - \frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right)\right)^{-\beta}.$$

Moreover,

$$\frac{\partial}{\partial s_l}\left(\frac{1}{1-(1-\beta+\beta s_l)\frac{s^f}{s_l}} - \frac{1}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}\right) = \frac{1-\beta}{s_l^2}\left(\frac{\overline{s}}{\left(1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}\right)^2} - \frac{s^f}{\left(1-(1-\beta+\beta s_l)\frac{s^f}{s_l}\right)^2}\right),$$

which is strictly positive since $\overline{s} > s^f$. It follows that E_n is decreasing in s_l. Since E_b does not depend on s_l, this implies that E_b/E_n is increasing in s_l. □

VIII.6.7 Proof of Lemma XXIV.

Proof. Note that $E_b/E_n = (\Psi(s_l,\overline{s}))^{\frac{\widetilde{\alpha}\beta}{1-\widetilde{\alpha}}}$, where

$$\Psi(s_l,\overline{s}) \equiv \frac{1-\widetilde{\alpha}(1-\beta+\beta\overline{s})}{1-\overline{s}} \frac{1-\frac{\overline{s}}{N}}{1-\widetilde{\alpha}\left(1-\beta+\beta\frac{\overline{s}}{N}\right)} \frac{1-(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}}{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{s_l}} \frac{1-\widetilde{\alpha}(1-\beta+\beta s_l)\frac{\overline{s}}{N s_l}}{1-(1-\beta+\beta s_l)\frac{\overline{s}}{N s_l}}.$$

Note that $E_b/E_n > 1$ if $\Psi(s_l,\overline{s}) > 1$, and $E_b/E_n < 1$ if $\Psi(s_l,\overline{s}) < 1$. Routine but tedious calculations show that $\Psi(\overline{s},\overline{s}) < 1$, $\Psi(1,\overline{s}) > 1$, and $\partial\Psi/\partial s_l > 0$, which proves the lemma. □

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